

## 1 Markov Chain Basics

A Markov chain is a sequence of random variables  $X_n$ ,  $n = 0, 1, 2, \dots$ . Here is one interpretation of a Markov chain:  $X_n$  is the state of a particle at time  $n$ . At each time step, the particle can jump to another state. Formally, a Markov chain satisfies the Markov property:

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i), \quad (1)$$

for all  $n$ , and for all sequences of states  $i_0, \dots, i_{n-1}, i, j$ . In other words, the Markov chain does not have any memory; the transition probability only depends on the current state, and not the history of states that have been visited in the past.

(a) In lecture, we learned that we can specify Markov chains by providing three ingredients:  $\mathcal{X}$ ,  $P$ , and  $\pi_0$ . What do these represent, and what properties must they satisfy?

(b) If we specify  $\mathcal{X}$ ,  $P$ , and  $\pi_0$ , we are implicitly defining a sequence of random variables  $X_n$ ,  $n = 0, 1, 2, \dots$ , that satisfies (1). Explain why this is true.

(c) Calculate  $\mathbb{P}(X_1 = j)$  in terms of  $\pi_0$  and  $P$ . Then, express your answer in matrix notation. What is the formula for  $\mathbb{P}(X_n = j)$  in matrix form?

## 2 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be  $\mathcal{X} = \{1, 2, 3, 4, 5\}$ . State 3 represents the target, while states 4 and 5 indicate that you have overshoot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of  $\{3\}$  before  $\{4, 5\}$ .

### 3 Consecutive Flips

Suppose you are flipping a fair coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

(a) Construct an Markov chain that describes the situation with a start state and end state.

(b) Given that you have flipped a (Tails, Heads) so far, what is the expected number of flips to see the same side three times?

(c) What is the expected number of flips to see the same side three times, beginning at the start state?