CS 70 Discrete Mathematics and Probability Theory
Summer 2021

HW 1

Due: Sunday 7/4, 10:00 PM
Grace period until Sunday 7/4, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Fibonacci Proof

Let F_i be the i^{th} Fibonacci number, defined by $F_{i+2} = F_{i+1} + F_i$ and $F_0 = 0$, $F_1 = 1$. Prove that

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \dots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{(2^n - 1)}$ using induction.

(c) Why does the hypothesis in part (b) imply the conclusion from part (a)?

3 Functional Composition

For the following questions, let X, Y, and Z be sets.

(a) Suppose that $f : X \to Y$, $g : Y \to Z$, and $h : Z \to X$ are functions such that the composition $h(g(f(\cdot)))$ is a bijection. For each of the following statements, either prove that the statement is true, or provide a counterexample.

   (i) The function f is injective.

   (ii) The function g is bijective.

   (iii) The function h is surjective.

(b) Suppose that $p : X \to X$ is a function such that $p(p(p(x))) = x$ for all $x \in X$. Prove that p is a bijection.

4 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-decreasing. That is, $f(x) \le f(y)$ whenever $x \le y$.

(b) The set of all functions f from $\mathbb{N}$ to $\mathbb{N}$ such that f is non-increasing. That is, $f(x) \ge f(y)$ whenever $x \le y$.

(c) The set of all bijective functions from $\mathbb{N}$ to $\mathbb{N}$.

5 Counting Cartesian Products

For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \dots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

6 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L to a vertex in R (so there does not exist an edge that connects two vertices in L or two vertices in R).

(a) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Prove that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$.

(b) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Let s and t denote the average degree of vertices in L and R respectively. Prove that $s/t = |R|/|L|$.

7 Doubled Graphs

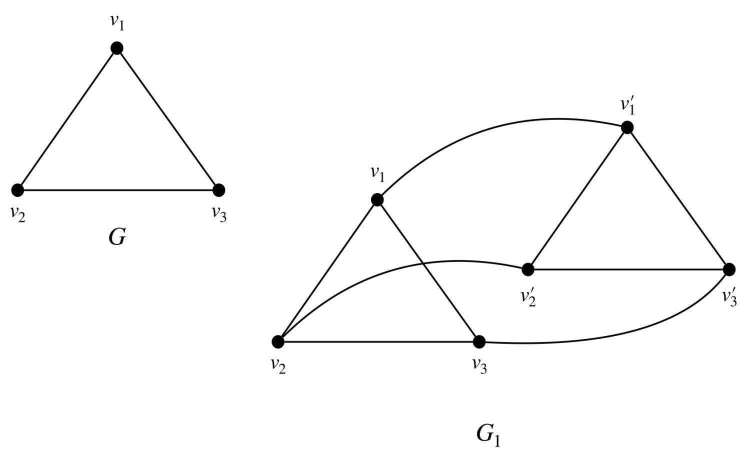
The *double* of a graph G consists of two copies of G with edges joining the corresponding vertices. More precisely, if $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices and E the set of edges, then the double of the graph G is given by $G_1 = (V_1, E_1)$, where

$$V_1 = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\},$$

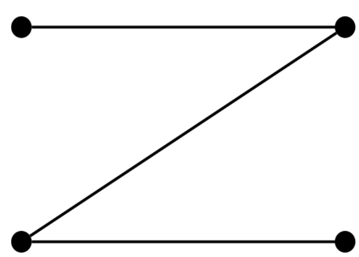
and

$$E_1 = E \cup \{(v'_i, v'_j) | (v_i, v_j) \in E\} \cup \{(v_i, v'_i), \forall i\}.$$

Here is an example,



(a) Draw the double of the following graph:



(b) Now suppose that G_1 is an arbitrary bipartite graph (see Problem 6), and that for each integer $n > 0$, we define G_{n+1} as the double of G_n. Show that $\forall n \geq 1$, G_n is bipartite.

Hint: Use induction on n.

8 Binary Trees

You may have seen the recursive definition of binary trees from previous classes. Here, we define binary trees in graph theoretic terms as follows (**Note:** here we will modify the definition of leaves slightly for consistency).

- A binary tree of height > 0 is a tree where exactly one vertex, called the **root**, has degree 2, and all other vertices have degrees 1 or 3. Each vertex of degree 1 is called a **leaf**. The **height** h is defined as the maximum length of the path between the root and any leaf.
- A binary tree of height 0 is the graph with a single vertex. The vertex is both a leaf and a root.

(a) Let T be a binary tree of height > 0, and let $h(T)$ denote it's height. Let r be the root in T and u and v be it's neighbors. Show that removing r from T will result in two binary trees, L, R with roots u and v respectively. Also, show that $h(T) = \max(h(L), h(R)) + 1$.

(b) Using the graph theoretic definition of binary trees, prove that the number of vertices in a binary tree of height h is at most $2^{h+1} - 1$.

(c) Prove that all binary trees with n leaves have $2n - 1$ vertices.