Due: Sunday 7/25, 10:00 PM
Grace period until Sunday 7/25, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Probability Potpourri

Provide brief justification for each part.

(a) For two events A and B in any probability space, show that $\mathbb{P}(A \setminus B) \geq \mathbb{P}(A) - \mathbb{P}(B)$.

(b) Suppose $\mathbb{P}(D \mid C) = \mathbb{P}(D \mid \overline{C})$, where $\overline{C}$ is the complement of C. Prove that D is independent of C.

(c) If A and B are disjoint, does that imply they're independent?

2 Conditional Practice

(a) Suppose you have 3 bags. Two of them contain a single \$10 bill, and the third contains a single \$5 bill. Suppose you pick one of these bags uniformly at random. You then add a \$5 bill to the bag, so it now contains two bills. The bag is shaken, and you randomly draw a bill from the bag without looking into the bag. Suppose it turns out to be a \$5 bill. If a you draw the remaining bill from the bag, what is the probability that it, too, is a \$5 bill? Show your calculations.

(b) Now suppose that you have a large number of bags, and that each of them contain either a gold or a silver coin (every bag contains exactly one coin). Moreover, these bags are either colored red, blue, or purple (every bag is exactly one of these colors). Half of the bags are red and a third of the bags are blue. Moreover, two thirds of the red bags and one fourth of the blue bags contain gold coins. Lastly, a randomly chosen bag has a $\frac{1}{2}$ probability of containing a silver coin. Suppose that you pick a bag at random and find that it contains a silver coin. What is the probability that the bag you picked was purple?

3 Playing Strategically

Bob, Eve and Carol bought new slingshots. Bob is not very accurate, hitting his target with probability $1/3$. Eve is better, hitting her target with probability $2/3$. Carol never misses. They decide to play the following game: They take turns shooting each other. For the game to be fair, Bob starts first, then Eve and finally Carol. Any player who gets shot has to leave the game. The last person standing wins the game. In this problem, we will investigate what Bob's best course of action would be.

(a) Compute the probability of the event E_1 that Bob wins in a duel against Eve alone, assuming he shoots first. (Hint: Let x be the probability Bob wins in a duel against Eve alone, assuming he fires first. If Bob misses his first shot and then Eve misses her first shot, what is the probability Bob wins in terms of x?)

(b) Compute the probability of the event E_2 that Bob wins in a duel against Eve alone, assuming he shoots second.

(c) Compute the probability of the same events for a duel of Bob against Carol.

(d) Assuming that both Eve and Carol play rationally, conclude that Bob's best course of action is to shoot into the air (i.e., intentionally miss)! (Hint: What happens if Bob misses? What if he doesn't?)

4 College Applications

There are n students applying to n colleges. Each college has a ranking over all students (i.e. a permutation) which, for all we know, is completely random and independent of other colleges.

College number i will admit the first k_i students in its ranking. If a student is not admitted to any college, he or she might file a complaint against the board of colleges, and colleges want to avoid that as much as possible.

(a) If for all i, $k_i = 1$ (i.e. if every college only admits the top student on its list), what is the probability that all students will be admitted to at least one college?

(b) What is the probability that a particular student, Alice, does not get admitted to any college? Prove that if the average of all k_i's is at least $2\ln n$, then this probability is at most $1/n^2$. (Hint: use the inequality $1 - x \leq e^{-x}$)

(c) Prove that when the average k_i is at least $2\ln n$, then the probability that at least one student does not get admitted to any college is at most $1/n$.

5 (Un)conditional (In)equalities

Let us consider a sample space $\Omega = \{\omega_1, \dots, \omega_N\}$ of size $N > 2$ and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

(a) Suppose that for every subset $A \subset \Omega$ of size $|A| = 2$ and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A)$. Is it necessarily true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)

(b) Suppose that for every subset $A \subset \Omega$ of size $|A| = k$, where k is some fixed element in $\{2, \ldots, N\}$, and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1(\omega \mid A) = \mathbb{P}_2(\omega \mid A)$. Is it necessarily true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$?

For the following two parts, assume that $\Omega = \left\{(a_1, \ldots, a_k) \mid \sum_{j=1}^{k} a_j = n\right\}$ is the set of configurations of n balls into k labeled bins, and let $\mathbb{P}_1$ be the probabilities assigned to these configurations by throwing the balls independently one after another into the bins, and let $\mathbb{P}_2$ be the probabilities assigned to these configurations by uniformly sampling one of these configurations.

(c) Let A be the event that all n balls are in exactly one bin.

   (i) What are $\mathbb{P}_1(\omega \mid A)$ and $\mathbb{P}_2(\omega \mid A)$ for any $\omega \in A$?

   (ii) Repeat part (i) for $\omega \in \Omega \setminus A$.

   (iii) Is it true that $\mathbb{P}_1(\omega) = \mathbb{P}_2(\omega)$ for all $\omega \in \Omega$?

(d) For the special case of $n = 9$ and $k = 3$, provide two outcomes B and C, so that $\mathbb{P}_1(B) < \mathbb{P}_2(B)$ and $\mathbb{P}_1(C) > \mathbb{P}_2(C)$. Provide justification.

6 Cliques in Random Graphs

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. Let's call the event that S forms a clique E_S. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \ldots, v_\ell\}$ and $V_2 = \{w_1, \ldots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a k-clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$.

7 Socks

Suppose you have n different pairs of socks (n left socks and n right socks, for $2n$ individual socks total) in your dresser. You take the socks out of the dresser one by one without looking and lay them out in a row on the floor. What is the probability that no two matching socks are next to each other?

8 Minesweeper

Minesweeper is a game that takes place on a grid of squares. When you click a square, it reveals either an integer $\in [1,8]$, a mine, or a blank space. If it reveals a mine, you instantly lose. If it reveals a number, that number refers to the number of mines adjacent to that square (including diagonally adjacent). If it reveals a blank space, there were 0 mines adjacent to it.

You are playing on a 8×8 board with 10 mines randomly distributed across the board. In your first move, you click a square near the center of the board (i.e. you click a square that is neither a corner square nor an edge square).

(a) What is the probability that the square reveals

  i. a mine?

  ii. a blank space?

  iii. the number k?

(b) Suppose the first square you clicked revealed the number k. For your next move, you want to minimize the probability of picking a mine. Should you click a square adjacent to your first pick, or a different square? Your answer should depend on the value of k.

(c) Now suppose the first square you clicked revealed the number 1. You then click the square to the right for your next move. What is the probability that this square reveals the number 4?