CS 70   Discrete Mathematics and Probability Theory
Summer 2021

HW 5

Due: Sunday 8/1, 10:00 PM
Grace period until Sunday 8/1, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Throwing Frisbees

Shahzar and his $n - 1$ friends stand in a circle and play the following game: Shahzar throws a frisbee to one of the other people in the circle randomly, with each person being equally likely, and thereafter, the person holding the frisbee throws it to someone else in the circle, again uniformly at random. The game ends when someone throws the frisbee back to Shahzar.

(a) What is the expected number of times the frisbee is thrown through the course of the game?

(b) What is the expected number of people that never get the frisbee during the game?

2 Various Variance Problems

(a) Suppose that X and Y are both binomial random variables with parameters n and p and $\text{Var}(X - Y) = 2$. Find $\text{cov}(X, Y)$ in terms of n and p.

(b) Prove that if X and Y are independent random variables, then

$$\text{Var}(XY) = \text{Var}(X)\,\text{Var}(Y) + \mathbb{E}[X]^2\,\text{Var}(Y) + \mathbb{E}[Y]^2\,\text{Var}(X).$$

3 Testing Model Planes

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying any remaining model planes and throwing away any that crash. Let X_i be the random variable

representing how many model planes remain after i days. Note that $X_0 = n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

4 Unreliable Servers

In a single cluster of a Google competitor, there are a huge number of servers n, each with a uniform and independent probability of going down in a given day. On average, 4 servers go down in the cluster per day. As each cluster is responsible for a huge amount of internet traffic, it is fair to assume that n is a very large number. Recall that as $n \to \infty$, a Binom$(n, \lambda/n)$ distribution will tend towards a Poisson(λ) distribution.

(a) What is an appropriate distribution to model the number of servers that crash on any given day for a certain cluster? What is its parameter?

(b) Compute the expected value and variance of the number of crashed servers on a given day for a certain cluster.

(c) Compute the probability that fewer than 3 servers crashed on a given day for a certain cluster.

(d) Compute the probability at least 3 servers crashed on a given day for a certain cluster.

5 Short Answer

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 2X + 1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + 1/4 & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c. Are X and Y independent?

(c) Let $X \sim \text{Exp}(3)$.

    (i) Find probability that $X \in [0, 1]$.

    (ii) Let $Y = \lfloor X \rfloor$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1,...,n$ be mutually independent. It is a (very nice) fact that $\min(X_1,...,X_n) \sim \text{Exp}(\mu)$. Find μ.

6 Arrows

You and your friend are competing in an archery competition. You are a more skilled archer than he is, and the distances of your arrows to the center of the bullseye are i.i.d. Uniform$[0,1]$ whereas his are i.i.d. Uniform$[0,2]$. To even out the playing field, you both agree that you will shoot one arrow and he will shoot two. The arrow closest to the center of the bullseye wins the competition. What is the probability that you will win? *Note*: The distances *from the center of the bullseye* are uniform.

7 Waiting For the Bus

Edward and Jerry are waiting at the bus stop outside of Soda Hall.

Like many bus systems, buses arrive in periodic intervals. However, the Berkeley bus system is unreliable, so the length of these intervals are random, and follow Exponential distributions.

Edward is waiting for the 51B, which arrives according to an Exponential distribution with parameter λ. That is, if we let the random variable X_i correspond to the difference between the arrival time ith and $i - 1$st bus (also known as the inter-arrival time) of the 51B, $X_i \sim \text{Expo}(\lambda)$.

Jerry is waiting for the 79, whose inter-arrival times also follows Exponential distributions with parameter μ. That is, if we let Y_i denote the inter-arrival time of the 79, $Y_i \sim \text{Expo}(\mu)$. Assume that all inter-arrival times are independent.

(a) What is the probability that Jerry's bus arrives before Edward's bus?

(b) After 20 minutes, the 79 arrives, and Jerry rides the bus. However, the 51B still hasn't arrived yet. Let D be the additional amount of time Edward needs to wait for the 51B to arrive. What is the distribution of D?

(c) Lavanya isn't picky, so she will wait until either the 51B or the 79 bus arrives. Find the distribution of Z, the amount of time Lavanya will wait before catching her bus.

(d) Khalil doesn't feel like riding the bus with Edward. He decides that he will wait for the second arrival of the 51B to ride the bus. Find the distribution of $T = X_1 + X_2$, the amount of time that Khalil will wait to ride the bus.

8 Variance of the Minimum of Uniform Random Variables

Let n be a positive integer and let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0, 1]$. Find $\text{Var}[Y]$, where

$$Y := \min\{X_1, \ldots, X_n\}.$$

Hint: You may need to perform integration by parts.