

Due: Tuesday 8/10, 10:00 PM
Grace period until Tuesday 8/10, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 A Bit of Everything

Suppose that X_0, X_1, \dots is a Markov chain with finite state space $S = \{1, 2, \dots, n\}$, where $n > 2$, and transition matrix P . Suppose further that

$$P(1, i) = \frac{1}{n} \quad \text{for all states } i \text{ and}$$
$$P(j, j-1) = 1 \quad \text{for all states } j \neq 1,$$

with $P(i, j) = 0$ everywhere else.

- Prove that this Markov chain is irreducible and aperiodic.
- Suppose you start at state 1. What is the distribution of T , where T is the number of transitions until you leave state 1 for the first time?
- Again starting from state 1, what is the expected number of transitions until you reach state n for the first time?
- Again starting from state 1, what is the probability you reach state n before you reach state 2?
- Compute the stationary distribution of this Markov chain.
- Suppose now you start in state n . What is the expected number of transitions until you return to state n for the first time?

2 Boba in a Straw

Imagine that Jonathan is drinking milk tea and he has a very short straw: it has enough room to fit two boba (see figure).

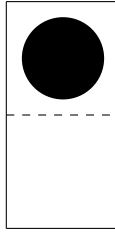


Figure 1: A straw with one boba currently inside. The straw only has enough room to fit two boba.

Here is a formal description of the drinking process: We model the straw as having two “components” (the top component and the bottom component). At any given time, a component can contain nothing, or one boba. As Jonathan drinks from the straw, the following happens every second:

1. The contents of the top component enter Jonathan’s mouth.
2. The contents of the bottom component move to the top component.
3. With probability p , a new boba enters the bottom component; otherwise the bottom component is now empty.

Help Jonathan evaluate the consequences of his incessant drinking!

- (a) Draw the Markov chain that models this process, and show that it is both irreducible and aperiodic.
- (b) At the very start, the straw starts off completely empty. What is the expected number of seconds that elapse before the straw is completely filled with boba for the first time? [Write down the equations; you do not have to solve them.]
- (c) Consider a slight variant of the previous part: now the straw is narrower at the bottom than at the top. This affects the drinking speed: if either (i) a new boba is about to enter the bottom component or (ii) a boba from the bottom component is about to move to the top component, then the action takes two seconds. If both (i) and (ii) are about to happen, then the action takes three seconds. Otherwise, the action takes one second. Under these conditions, answer the previous part again. [Write down the equations; you do not have to solve them.]
- (d) Jonathan was annoyed by the straw so he bought a fresh new straw (same as the straw from Figure 1). What is the long-run average rate of Jonathan’s calorie consumption? (Each boba is roughly 10 calories.)
- (e) What is the long-run average number of boba which can be found inside the straw? [Maybe you should first think about the long-run distribution of the number of boba.]
- (f) What is the long run probability that the amount of boba in the straw doesn’t change from one second to the next?