① Goals of this class

- Learn mathematical tools useful for CS

  Modular arithmetic, polynomials, probability

- Learn to think like a mathematician

  Understand mathematical notation and language
  Read and write valid proofs
  Problem solving

Today: the "grammar" of math

② An Example: Basic Arithmetic

(I promise this is relevant)

How to count to 10 in Basque

1	bat
2	bi
3	hiru
4	lau
5	bost
6	sei
7	zazpi
8	zortzi
9	bederatzi
10	hamar

Take a few seconds to look over the list

Question What is hiru + lau?

Question Count backwards from hamar to bat

Not that easy!

The point: Even things you now think of as simple take time & practice to master.

Optimistic: Even if things in this class seem hard or confusing, you can master them with practice

Pessimistic: You will need to put in time & effort

Just watching lecture = Trying to count in Basque after looking at the previous slide for 10 seconds

③ My advice

- See all the material multiple times

  Skim notes before lecture
  Attend lecture
  Review notes after lecture

- Give yourself plenty of time to think about homework problems

  Monday morning: Old hw already due, no new lectures yet. Why not take a look at the homework problems?

④ Propositional Logic

Onward to the actual content!

Def A proposition is a sentence that asserts a fact which is either true or false, but not both

Example

$1+1=3$ false proposition

Sacramento is the capital of California. true proposition

$x+x=y$ not a proposition. what are x and y?

Please don't read this sentence. not a proposition
does not assert a fact

Def The truth value of a proposition is true (T) if the proposition is true and false (F) if it is false

4.1 Combining Propositions

We can form new propositions from old ones

Example: Sacramento is the capital of California and $1+1=3$.

Use letters to denote propositional variables
P, Q, R

Def If P and Q are propositions:

operation	symbols	meaning
Negation	$\neg P$	"It is not the case that P"
Conjunction	$P \wedge Q$	"P and Q"
Disjunction	$P \vee Q$	"Either P or Q or both"

Example: P = "I am a carrot" Q = "I am an avocado"
$P \wedge \neg Q$ = "I am a carrot and I am not an avocado"

4.2 Truth Tables

A way to systematically record what an operation on propositions is doing

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$\neg(P \vee Q)$
T	T	F	T	T	F
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	F	T

we can also use truth tables for more complicated formulas

4.3 More ways of combining propositions

Def If P and Q are propositions:

Implication $P \Rightarrow Q$ "If P, then Q" (hypothesis: P; conclusion: Q)

Bi-implication $P \Leftrightarrow Q$ "P if and only if Q" (iff)

P	Q	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

A false proposition implies everything!

Warning!! Mathematicians use implication to mean something a little different from the everyday meaning

Example If Berkeley is the capital of California, then I am a carrot. ← true proposition!

4.4 More about implication

Other ways to express $P \Rightarrow Q$ in English.

P = "You like CS 70"
Q = "You like probability"

If you like CS 70 then you like probability.
You like CS 70 only if you like probability.
You like probability if you like CS 70.
The fact that you like probability is a necessary condition for you to like CS 70.

Def Given an implication $P \Rightarrow Q$.
The converse is the implication $Q \Rightarrow P$
The contrapositive is the implication $\neg Q \Rightarrow \neg P$
The inverse is the implication $\neg P \Rightarrow \neg Q$

⑤ Logical Equivalence

Def A propositional formula is an expression made up of propositional variables combined with operations like $\neg, \wedge, \vee, \Rightarrow$ = compound proposition

Example $(P \vee (\neg Q \Rightarrow (R \wedge P))) \Leftrightarrow \neg(P \wedge R)$

Def A propositional formula is a tautology if it is true no matter what the truth values of its propositional variables are

Example $P \vee \neg P$ ↳ tautology

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

→ all true

$P \vee Q$ ↳ not a tautology

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Goal of math: Find interesting tautologies

Def Two propositional formulas are **logically equivalent** if they have matching truth values for any setting of the propositional variables

Example $P \Rightarrow Q$ and $\neg P \vee Q$

P	Q	$\neg P$	$P \Rightarrow Q$	$\neg P \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

→ the two columns are the same

$(P \Rightarrow Q) \equiv (\neg P \vee Q)$

→ logical equivalence

5.1 Some ~~interesting~~ useful examples of logical equivalence

Question Which of the following is logically equivalent to $P \Rightarrow Q$?

- Converse $(Q \Rightarrow P)$ not equivalent
- Contrapositive $(\neg Q \Rightarrow \neg P)$ equivalent

Answer:

P	Q	$\neg Q$	$\neg P$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T	T	T
T	F	T	F	F	T	F
F	T	F	T	T	F	T
F	F	T	T	T	T	T

Example If you like CS70, then you like probability.

$\equiv$ If you don't like probability, then you don't like CS 70.

$\not\equiv$ If you like probability, then you like CS 70.

↳ what if you don't like discrete math?

More examples

$$\left.\begin{aligned} \neg(P \wedge Q) &\equiv (\neg P \vee \neg Q) \\ \neg(P \vee Q) &\equiv (\neg P \wedge \neg Q) \end{aligned}\right\} \text{De Morgan's laws}$$

⑥ Quantifiers

We can write more interesting statements using quantifiers.

Recall $x+x=y$ is not a proposition

But it is if we plug in values for x and y

E.g. $3+3=17$

Def A **propositional function** is a statement containing variables which becomes a proposition once we plug in values for the variables

Example Let $P(x,y)$ be the statement "$x+x=y$"

"where x and y can be any natural numbers"

$P(1,2)$ is true

$P(3,17)$ is false

Def The **domain** or **universe** of a propositional function $P(x)$ is the set of possible values of x

The universe should be specified when $P(x)$ is defined

Def If $P(x)$ is a propositional function then

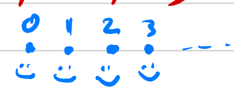
$\forall x\, P(x)$ denotes the proposition "$P(x)$ for each value of x in the domain"

$\exists x\, P(x)$ denotes the proposition "there exists at least one element x in the domain such that $P(x)$"

Write $\forall x \in S\; P(x)$ or $\exists x \in S\; P(x)$ to indicate that the domain is S

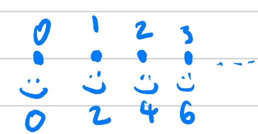
natural numbers $\{0, 1, 2, 3, \dots\}$

Example $\forall x \in \mathbb{N},\; x \geq 0$ true



$\exists x \in \mathbb{N},\; x \geq 0$ true

$\forall x \in \mathbb{N}\; \exists y \in \mathbb{N},\; x + x = y$ true



Example "Every nonzero rational number can be multiplied by some rational number to get 1"

rational numbers ($1/2$, $-17/32$, ...)

$$\forall x \in \mathbb{Q} \; (x \neq 0 \Rightarrow (\exists y \in \mathbb{Q},\; x \cdot y = 1))$$

"There are at least 2 integers"

$$\exists x \in \mathbb{Z} \; \exists y \in \mathbb{Z} \; (x \neq y)$$

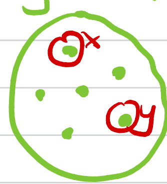
integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

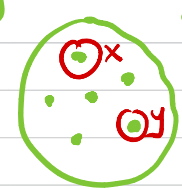
6.1 Logical Equivalence with Quantifiers

Def Two propositions built up out of propositional functions, $\neg$, $\wedge$, $\vee$, $\to$ and quantifiers are logically equivalent if they have the same truth value no matter what the propositional functions are.

Can't use truth tables for this

Example $\exists x \exists y\, P(x,y) \equiv \exists y \exists x\, P(x,y)$





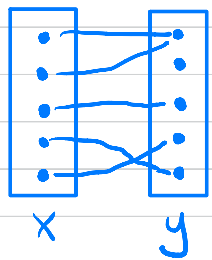
De Morgan's laws for quantifiers

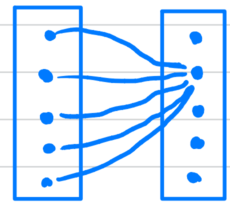
$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

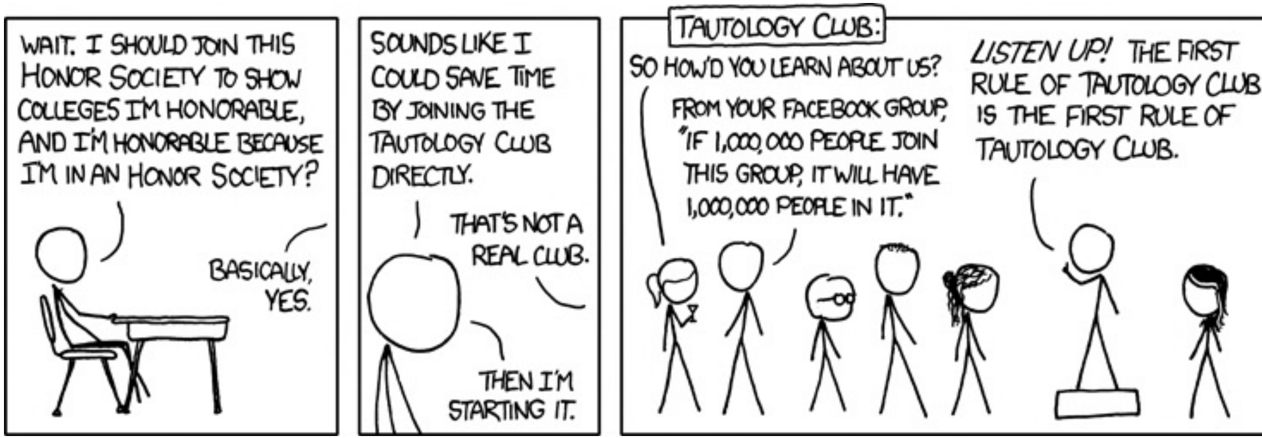
Example Are $\forall x \exists y\, P(x,y)$ and $\exists y \forall x\, P(x,y)$ logically equivalent?

No.





"Everybody has a mother" vs. "There is someone who is everybody's mother."


WAIT. I SHOULD JOIN THIS HONOR SOCIETY TO SHOW COLLEGES I'M HONORABLE, AND I'M HONORABLE BECAUSE I'M IN AN HONOR SOCIETY?
BASICALLY, YES.
SOUNDS LIKE I COULD SAVE TIME BY JOINING THE TAUTOLOGY CLUB DIRECTLY.
THAT'S NOT A REAL CLUB.
THEN I'M STARTING IT.
TAUTOLOGY CLUB:
SO HOW'D YOU LEARN ABOUT US?
FROM YOUR FACEBOOK GROUP, "IF 1,000,000 PEOPLE JOIN THIS GROUP, IT WILL HAVE 1,000,000 PEOPLE IN IT."
LISTEN UP! THE FIRST RULE OF TAUTOLOGY CLUB IS THE FIRST RULE OF TAUTOLOGY CLUB.

xkcd #703