0 How to read like a mathematician

<u>Maxim #1</u> When you read mathematical writing and you come across a proof

<u>Maxim #2</u> At the heart of most proofs, definitions, etc, there is a simple idea.

① Induction

The sledgehammer of math 

$P(n)$: propositional function with domain $\mathbb{N}$

I tell you :

Question

Principle of induction To prove , it is enough to prove:

Base Case:

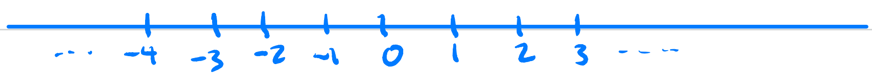
Inductive Step:

Principle of induction To prove $\forall n \in \mathbb{N}\ P(n)$, it is enough to prove:

Base Case: $P(0)$

Inductive Step: $\forall n \in \mathbb{N}\ (P(n) \Rightarrow P(n+1))$

Question Would this work if we replaced $\mathbb{N}$ with $\mathbb{Z}$?



② Examples of using induction

Question What is $0+1+2+\dots+n$?

Claim

How to prove it?

Thm For all $n \in \mathbb{N}$, $0+1+\ldots+n = \frac{n(n+1)}{2}$

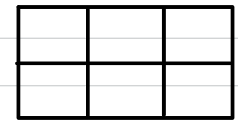
proof Base Case:

Inductive hypothesis:

Inductive step:

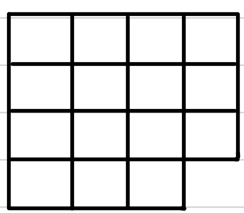
(2.1) Another example

Triomino tile:

Tiling a grid with triominos: 

Question Can you tile a chessboard with triominos?

Easier version:



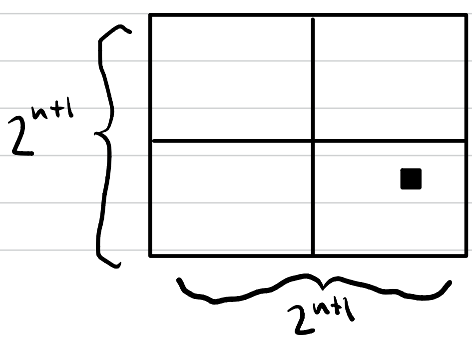
Thm

Thm For every $n \in \mathbb{N}$, any $2^n \times 2^n$ grid with one square removed can be tiled with triominos

proof **Base case:**

Inductive hypothesis: Assume

Inductive step: Have a $2^{n+1} \times 2^{n+1}$ grid with one square removed. Want to show



1

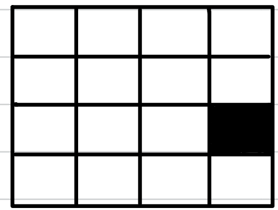
2

3

4

Question Tile this grid with triominos

The induction argument tells you how!



③ When is induction useful?

When you want to prove a statement about all natural numbers

Or all natural numbers in some range

Example

Especially if

1.

   Example

2.

(4) Variations on induction

(1) Different base case

(2) Multiple base cases

(3) Change the inductive assumption

(4) Strong induction

(4.1) Changing the inductive hypothesis

Thm For every $n \in \mathbb{N}$,

proof attempt

Base case:

Inductive hypothesis:

Inductive step:

Thm For every $n \in \mathbb{N}$, the sum of the first n odd numbers is a perfect square

It would help to know what the sum of the first n odd numbers is equal to

proof We will show

Base Case:

Inductive hypothesis:

Inductive step:

4.2 Strong induction

Regular induction

Strong induction

<u>Thm</u> Every natural number

<u>Example</u>

<u>proof attempt</u> <u>Base case:</u>

<u>Inductive hypothesis:</u>

<u>Inductive step:</u>

Two cases

<u>Case 1:</u>

<u>Case 2:</u>

Thm Every natural number $n > 1$ is a product of prime numbers

Example $12 = 2 \cdot 2 \cdot 3$ $17 = 17$ $57 = 3 \cdot 19$ $60 = 15 \cdot 4$
$= 5 \cdot 3 \cdot 2 \cdot 2$

proof attempt **Base case:** $n = 2$
2 is prime ✓

Inductive hypothesis: Assume $n > 1$ and

Inductive step: WTS $n+1$ is a product of primes

Two cases

Case 1: $n+1$ is prime ✓

Case 2: $n+1$ is not prime
$\exists a, b,\ (n+1) = a \cdot b$ and $a, b \neq 1,\ a, b \neq n+1$

4.3 Aside: Induction, Strong induction, well-ordering

Not very important to understand

Induction

Strong induction

Well-ordering

Strong induction on $P(n)$ =

Induction on $P(n)$ from well ordering:

(4.4) Different base case & multiple base cases

Fibonacci sequence

How fast does the Fibonacci sequence grow?

n	
F_n	

It looks like

Thm

How to prove it?

Thm For all natural numbers $n \geq 6$, $F_n > n$

proof **Base cases:**

Inductive hypothesis:

Inductive step:

Exercise:

Harder:

⑤ Be Careful

Lecture 2 notes:

Thm

proof Let $P(n) =$

Base case:

Inductive hypothesis:

Inductive step:

Thm All people have the same number of hairs on their head (!?)

proof Let $P(n)$ = "in any set of n people, everyone has the same number of hairs on their head"

Show $P(n)$ by induction

Base case: $n=1$ 1 person ✓

Inductive hypothesis: Assume $P(n)$

Inductive step: $n+1$ people $a_1, a_2, \ldots, a_n, a_{n+1}$

IH $\Rightarrow$ $a_1, a_2, \ldots, a_n$ have same # of hairs

$a_2, a_3, \ldots, a_{n+1}$ have same # of hairs

So all have same # of hairs as a_2

What's wrong with this?