0 How to read like a mathematician

<u>Maxim #1</u> When you read mathematical writing and you come across a proof, cover it up and try to figure it out yourself

You may not succeed, but you will understand what obstacles the proof has to overcome

<u>Maxim #2</u> At the heart of most proofs, definitions, etc., there is a simple idea.

Your mission: find it!

① Induction

The sledgehammer of math

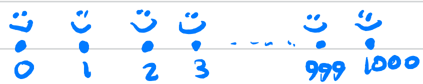
recall: for each $n \in \mathbb{N}$, $P(n)$ is a proposition

$P(n)$: propositional function with domain $\mathbb{N}$

I tell you: $P(0)$ is true

$P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$

Question: Is $P(1000)$ true? Yes!



$P(0)$
$P(0) \Rightarrow P(1)$
$P(1) \Rightarrow P(2)$
$\vdots$
$P(999) \Rightarrow P(1000)$

Principle of induction: To prove $\forall n \in \mathbb{N}\ P(n)$, it is enough to prove:

Base Case: $P(0)$

Inductive Step: $\forall n \in \mathbb{N}\ (P(n) \Rightarrow P(n+1))$

Principle of induction To prove $\forall n \in \mathbb{N}\ P(n)$, it is enough to prove:

Base Case: $P(0)$

Inductive Step: $\forall n \in \mathbb{N}\ (P(n) \Rightarrow P(n+1))$

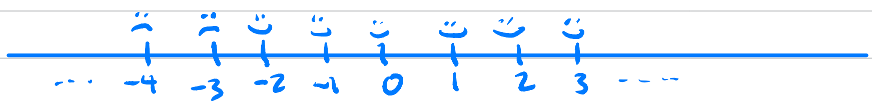
inductive hypothesis

Question Would this work if we replaced $\mathbb{N}$ with $\mathbb{Z}$? No!

Counterexample: $P(n) =$ "$n \geq -2$" with domain $\mathbb{Z}$

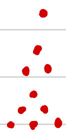
$P(0) =$ "$0 \geq -2$" ✓

$P(n) \Rightarrow P(n+1)$ "$n \geq -2 \Rightarrow n+1 \geq -2$" ✓



② Examples of using induction

Question What is $0+1+2+\dots+n$?



$$0 \qquad\qquad \frac{0(0+1)}{2}=0$$

$$0+1=1 \qquad\qquad \frac{1(1+1)}{2}=1$$

$$0+1+2=3 \qquad\qquad \frac{2(2+1)}{2}=3$$

$$0+1+2+3=6 \qquad\qquad \frac{3(3+1)}{2}=6$$

$$\vdots \qquad\qquad \vdots$$

triangular numbers

Claim $0+1+2+\dots+n=\dfrac{n(n+1)}{2}$

How to prove it? Induction!

$$= \sum_{i=0}^{n} i$$

Thm For all $n \in \mathbb{N}$, $0+1+\dots+n = \dfrac{n(n+1)}{2}$

proof Base Case: $n=0$. $0 = \dfrac{0(0+1)}{2}$

Inductive hypothesis: Assume $0+1+\dots+n = \dfrac{n(n+1)}{2}$

Inductive hypothesis

Inductive step: Use IH to show that

$$0+1+\dots+n+(n+1) = \frac{(n+1)((n+1)+1)}{2}$$

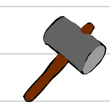
$$(0+1+\dots+n)+(n+1) = \frac{n(n+1)}{2}+(n+1) \quad \text{(By IH)}$$

$$= \frac{n(n+1)}{2}+\frac{2(n+1)}{2}$$

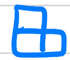
$$= \frac{n(n+1)+2(n+1)}{2}$$

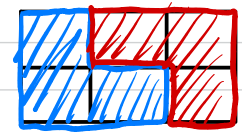
$$= \frac{(n+2)(n+1)}{2}$$

$$= \frac{(n+1)((n+1)+1)}{2}.$$



2.1 Another example

Triomino tile: 

Tiling a grid with triominos: 

rotations allowed

Tiling: cover all squares, no overlaps

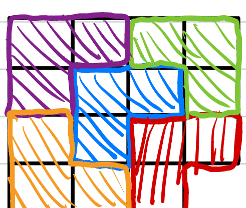
Question Can you tile a chessboard with triominos?

chessboard: 8×8 grid

No! Chessboard has $8 \cdot 8 = 64$ squares, not divisible by 3

What if we remove one square?

Easier version: tile a 4×4 grid with 1 square removed?



Thm For every $n \in \mathbb{N}$, any $2^n \times 2^n$ grid with one square removed can be tiled with triominos.

Thm For every $n \in \mathbb{N}$, any $2^n \times 2^n$ grid with one square removed can be tiled with triominos

proof **Base case:** $n=0$. $2^0 = 1$

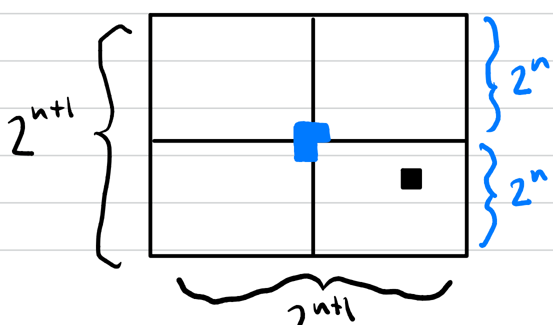
1×1 grid □

1 square removed

nothing left
easy to tile!

Inductive hypothesis: Assume all $2^n \times 2^n$ grids with one square removed can be tiled.

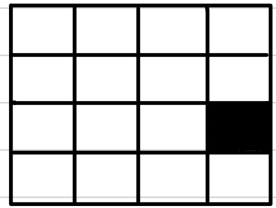
Inductive step: Have a $2^{n+1} \times 2^{n+1}$ grid with one square removed. Want to show it can be tiled.



① Divide into 4 $2^n \times 2^n$ grids
② Place triomino in center
③ Now each $2^n \times 2^n$ piece is missing one piece
④ Tile each one using IH

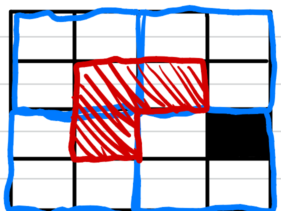
Question Tile this grid with triominos

The induction argument tells you how!

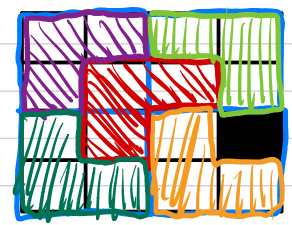


Answer

First divide into 4 2×2 grids & put a triomino in the center



Now recursively tile each 2×2 grid



Lesson: proofs by induction often tell you recursive algorithms

③ When is induction useful?

When you want to prove a statement about all natural numbers

Or all natural numbers in some range

Example $\forall n \in \mathbb{N}\ \left(0+1+2+\dots+n = \frac{n(n+1)}{2}\right)$

Especially if

① You are proving something about a recursively defined sequence

Example Fibonacci numbers

$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n$

$0, 1, 1, 2, 3, 5, 8, \dots$

② You are proving something about a recursive algorithm or an algorithm that loops

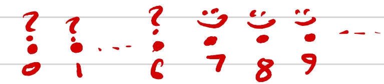
$\hookrightarrow$ CS 170!

④ Variations on induction

① Different base case

Prove $P(7)$ and $P(n) \Rightarrow P(n+1)$

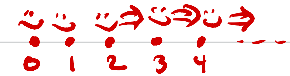
Conclude $\forall n \geq 7 \; P(n)$ $= \forall n \in \mathbb{N} \, (n \geq 7 \Rightarrow P(n))$



② Multiple base cases

Prove $P(0)$, $P(1)$, $P(2)$ and $\forall n \geq 2 \, (P(n) \Rightarrow P(n+1))$

Conclude $\forall n \in \mathbb{N} \; P(n)$



③ Change the inductive assumption

Want to prove $\forall n \in \mathbb{N} \; P(n)$

Find some $Q(n)$ such that $Q(n) \Rightarrow P(n)$ — Q is stronger than P

Prove $Q(0)$ and $Q(n) \Rightarrow Q(n+1)$ ← looks harder, but useful if $P(n) \Rightarrow P(n+1)$ hard to prove because P is too weak

④ Strong induction

Prove $P(0)$ and $(P(0) \wedge P(1) \wedge \dots \wedge P(n)) \Rightarrow P(n+1)$

Conclude $\forall n \in \mathbb{N} \; P(n)$ ← secretly just a special case of ③

4.1 Changing the inductive hypothesis

Thm For every $n \in \mathbb{N}$, the sum of the first n odd numbers is a perfect square → i.e. $= k^2$ for some $k \in \mathbb{N}$

proof attempt

Base case: $n = 1$

First odd number: 1

$1 = 1^2$ ✓

where $n \geq 1$

Inductive hypothesis: Sum of first n odd numbers is perfect square

want to show

Inductive step: WTS: Sum of first $n+1$ odd numbers is perfect square

$(n+1)^{st}$ odd number: $2n+1$

$$\begin{aligned} \text{sum of first } n+1 \text{ odd \#s} &= (\text{sum of first } n \text{ odd \#s}) + (2n+1) \\ &= k^2 + (2n+1) \quad \text{for some } k \in \mathbb{N} \\ &= ?? \end{aligned}$$

It would help to know what the sum of the first n odd numbers is equal to

Thm For every $n \in \mathbb{N}$, the sum of the first n odd numbers is a perfect square

It would help to know what the sum of the first n odd numbers is equal to

$1 = 1 = 1^2$
$1+3 = 4 = 2^2$
$1+3+5 = 9 = 3^2$

$1+3+5+7 = 16 = 4^2$
$\vdots$

It looks like sum of first n odd #s $= n^2$

proof We will show the sum of the first n odd #s is n^2

Base Case: $n=1$
$1 = 1^2$

actually, base case of $n=0$ also works

Inductive hypothesis: sum of first n odd #s is n^2

Inductive step: WTS sum of first $(n+1)$ odd #s is $(n+1)^2$

$$\begin{aligned} \text{sum of first } (n+1) \text{ odd \#s} &= (\text{sum of first } n) + (2n+1) \\ &= n^2 + (2n+1) \quad \text{(By IH)} \\ &= (n+1)^2 \end{aligned}$$

Lesson: when you get stuck, work out small examples

4.2 Strong induction

Regular induction

Prove $P(0)$

Prove $P(n) \Rightarrow P(n+1)$

Technically, $\forall n \in \mathbb{N}\, (P(n) \Rightarrow P(n+1))$

Conclude $\forall n \in \mathbb{N},\ P(n)$

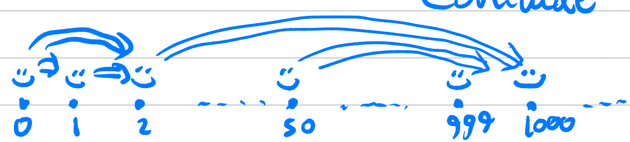
Strong induction

Prove $P(0)$

Prove $(\forall k \le n,\ P(k)) \Rightarrow P(n+1)$

$= P(0) \land P(1) \land \dots \land P(n)$

Conclude $\forall n \in \mathbb{N},\ P(n)$



Thm Every natural number $n > 1$ is a product of prime numbers

Example $12 = 2 \cdot 2 \cdot 3$ $\quad 17 = 17$ $\quad 57 = 3 \cdot 19$ $\quad 60 = 15 \cdot 4$
$= 5 \cdot 3 \cdot 2 \cdot 2$

proof attempt **Base case:** $n = 2$
2 is prime ✓

Inductive hypothesis: Assume $n > 1$ and n is a product of prime numbers ← need to replace this

Inductive step: WTS $n+1$ is a product of primes

Two cases

Case 1: $n+1$ is prime ✓

Case 2: $n+1$ is not prime

$\exists a, b,\ (n+1) = a \cdot b$ and $a, b \neq 1,\ a, b \neq n+1$

How can we use IH??

Done if we had IH for a and b!

Thm Every natural number $n > 1$ is a product of prime numbers

Example $12 = 2 \cdot 2 \cdot 3$ $\quad 17 = 17$ $\quad 57 = 3 \cdot 19$ $\quad 60 = 15 \cdot 4$
$= 5 \cdot 3 \cdot 2 \cdot 2$

proof attempt **Base case:** $n = 2$ ← Base case of $n=0$ / $n=1$ actually works
2 is prime ✓

Inductive hypothesis: Assume $n > 1$ and for all $k \leq n$, if $k > 1$ then k is a product of prime numbers

Inductive step: WTS $n+1$ is a product of primes

Two cases

Case 1: $n+1$ is prime ✓

Case 2: $n+1$ is not prime

$\exists a, b,\ (n+1) = a \cdot b$ and $a, b \neq 1$, $a, b \neq n+1$

$1 < a, b \leq n \Rightarrow$ a and b both product of primes

$\Rightarrow$ $n+1$ is a product of primes

IH

4.3 Aside: Induction, Strong induction, well-ordering

Not very important to understand

Induction $P(0)$ & $P(n) \Rightarrow P(n+1) \rightsquigarrow \forall n \in \mathbb{N}, P(n)$

Strong induction $P(0)$ & $(\forall k \leq n, P(k)) \Rightarrow P(n+1) \rightsquigarrow \forall n \in \mathbb{N}, P(n)$

Well-ordering All nonempty subsets of $\mathbb{N}$ have a least element

They are all equivalent

Strong induction on $P(n)$ = Regular induction on $Q(n) = \forall k \leq n, P(k)$

Induction on $P(n)$ from well ordering: Look at set of places where $P(n)$ doesn't hold

(4.4) Different base case & multiple base cases

Fibonacci sequence $F_0 = 0$ $F_1 = 1$ $F_{n+2} = F_n + F_{n+1}$

$F_0 = 0$
$F_1 = 1$
$F_2 = 0+1 = 1$
$F_3 = 1+1 = 2$
$F_4 = 1+2 = 3$
$F_5 = 2+3 = 5$
$F_6 = 3+5 = 8$
$\vdots$

How fast does the Fibonacci sequence grow?

Linear? Quadratic? Exponential? ...

n	0	1	2	3	4	5	6	7	8	9	...
F_n	0	1	1	2	3	5	8	13	21	34	...

It looks like eventually F_n is always bigger than n

Thm For all $n \geq 6$, $F_n > n$.

How to prove it? Induction! 

<u>Thm</u> For all natural numbers $n \geq 6$, $F_n > n$

<u>proof</u> <u>Base cases</u>: $n = 6$ $\quad F_6 = 8 > 6$

$n = 7$ $\quad F_7 = 13 > 7$

<u>Inductive hypothesis</u>: Assume $n \geq 7$ and for all $k \leq n$, if $k \geq 6$ then $F_k > k$

<u>Inductive step</u>: WTS $F_{n+1} > n+1$

$$\begin{aligned} F_{n+1} &= F_{n-1} + F_n \\ &> (n-1) + n \\ &= n+1 + (n-2) \\ &> n+1 \end{aligned}$$

By IH. Note $n \geq 7 \Rightarrow n, n-1 \geq 6$

$n \geq 7 \Rightarrow n-2 > 0$

<u>Exercise</u>: $F_n < 2^n$ for all n

<u>Harder</u>: Find an exact formula for F_n

⑤ Be Careful

Lecture 2 notes: There are at least 2 people in San Francisco who have the same # of hairs on their head

<u>Thm</u> All people have the same number of hairs on their head (!?)

<u>proof</u> Let $P(n)$ = "in any set of n people, everyone has the same number of hairs on their head"
Show $P(n)$ by induction

<u>Base case:</u> $n=1$ 1 person ✓

<u>Inductive hypothesis:</u> Assume $P(n)$

<u>Inductive step:</u> $n+1$ people $a_1, a_2, \ldots, a_n, a_{n+1}$

IH $\Rightarrow$ $a_1, a_2, \ldots, a_n$ have same # of hairs
$a_2, a_3, \ldots, a_{n+1}$ have same # of hairs
So all have same # of hairs as a_2

What's wrong with this?

Thm All people have the same number of hairs on their head (!?)

proof Let $P(n)$ = "in any set of n people, everyone has the same number of hairs on their head"

Show $P(n)$ by induction

Base case: $n=1$ 1 person ✓

Inductive hypothesis: Assume $P(n)$

Inductive step: $n+1$ people $a_1, a_2, \dots, a_n, a_{n+1}$

IH $\Rightarrow$ $a_1, a_2, \dots, a_n$ have same # of hairs

$a_2, a_3, \dots, a_{n+1}$ have same # of hairs

So all have same # of hairs as a_2

What's wrong with this?

Answer: Inductive step doesn't work when $n=1$.