⓪ Some advice on solving problems

Maxim #1 Don't give up after 30 seconds!

Common pattern:

Instead:

Maxim #2 Do the messy calculations/hard work!

Common pattern:

Instead:

① Introduction

<u>Question</u> Are there more real numbers than natural numbers?

<u>Goal for today</u>

Key step: finding a reasonable mathematical definition of

Not so important for CS. But an excuse to introduce useful mathematical terminology. Plus it's super cool!

② Sets

Def A set is an unordered collection of objects

Two sets are equal if they contain the same objects

Example $\{1,2,3\}$ $\{3,1,2\}$

$\{2,3,5\}$ set of all prime numbers < 6

$\{1,2,3\}$ $\{1,2,3,4\}$

Def $x \in A$ means x is an element of A

$x \notin A$ means x is not an element of A

Example 2 $\{1,2,3\}$, 0 $\{1,2,3\}$, $\sqrt{2}$ $\mathbb{R}$, $\sqrt{2}$ $\mathbb{Q}$

Def A set A is a subset of B, written $A \subseteq B$, if for every $x \in A$ we have $x \in B$ $\forall x\,(x \in A \Rightarrow x \in B)$

Example $\{1,2,3\}$ $\{1,2,3,4\}$, $\mathbb{N}$ $\mathbb{R}$, $\{1,2\}$ $\{1,2\}$, $\{-1\}$ $\mathbb{N}$

Question Suppose $A \subseteq B$ and $B \subseteq A$. Must $A = B$?

How to prove two sets are equal:

① Let x be an arbitrary element of A.

② Let x be an arbitrary element of B.

2.1 Some common sets

$\mathbb{N}$ natural numbers $\{0, 1, 2, \dots\}$

$\mathbb{Z}$ integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q}$ rational numbers $\frac{1}{2} \in \mathbb{Q}$, $\pi \notin \mathbb{Q}$

$\mathbb{R}$ real numbers $\sqrt{2} \in \mathbb{R}$, $\pi \in \mathbb{R}$

$\emptyset$ empty set

Question

1. $3 \in \mathbb{R}$?
2. $3 \in \emptyset$?
3. $\mathbb{N} \in \mathbb{R}$?
4. $\emptyset \subseteq \mathbb{N}$?
5. $\mathbb{N} \subseteq \emptyset$?
6. Steph Curry $\in \emptyset$?

(2.2) How to describe a set

(1) List the elements

(2) List enough elements to make it clear what set it is

$\{0, 2, 4, 6, \ldots\}$

$\{1, 2, 3, \ldots, 10\}$

Bad: $\{1, \ldots\}$

(3) Describe it in words

(4) Set builder notation

(a.k.a. set comprehension)

$\{n \in \mathbb{N} \mid \exists m \in \mathbb{N}\,(n = 2 \cdot m)\}$

$\{x \in \mathbb{R} \mid x^2 = -x\}$

$\{x \in \mathbb{N} \mid x^2 = -x\}$

(5) Some specialized notation. $a \leq b$ real numbers

$[a, b) =$

2.3 Operations on sets

1. **Union** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$\{1,2\} \cup \{1,7\} =$

$\{1,2,3\} \cup \emptyset =$

2. **Intersection** $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

$\{1,2\} \cap \{1,7\} =$

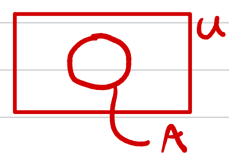
$\{1,2\} \cap \{2,3\} =$

3. **Difference** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

$\{1,2\} \setminus \{1,7\} =$

4. **Complement** Assume $A \subseteq U$. $A^c = U \setminus A$

$\mathbb{Q}^c =$



⑤ Cartesian product $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

$\{1,2\} \times \{2,3\} =$

Also $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1 \wedge \ldots \wedge a_n \in A_n\}$

<u>Example</u>

⑥ Powerset $\mathcal{P}(A)$ = set of all subsets of A

$\mathcal{P}(\{1,2\}) =$

<u>Question</u> $\emptyset \in \mathcal{P}(A)$ for any A?

$\mathbb{N} \in \mathcal{P}(\mathbb{R})$?

$A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$?

2.4 Cardinality of finite sets

Def If A is a finite set, write $|A|$ to denote the number of elements in A

$|\{-1, 7\}| =$

$|\{\text{Khalil, Shahzar, Patrick}\}| =$

Question If $|A| = |B| = 2$, what is $|A \times B|$?

$A = \{a_1, a_2\}$ $B = \{b_1, b_2\}$

$A \times B =$

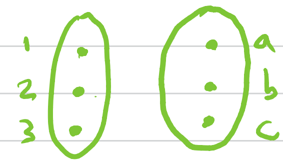
Exercise If $|A| = n$ and $|B| = k$ then $|A \times B| =$

If $|A| = n$ then $|\mathcal{P}(A)| =$

③ Functions

Def If A and B are sets, a **function** f from A to B is an assignment of exactly one element of B to each element of A

Example $f : \{1, 2, 3\} \to \{a, b, c\}$



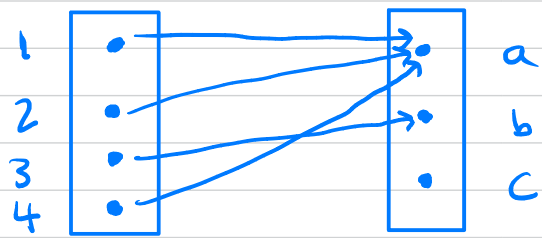
Def The **range** of a function $f : A \to B$, written **range(f)**, is the set

Example With f and g as above,

range(f) =
range(g) =

3.1 Anatomy of a function

$$f : A \longrightarrow B$$



$f(1) =$

3.2 How to describe functions

1. Explicitly list where each element of the domain is sent

$f: \{\text{Khalil}, \text{Shahzar}, \text{Patrick}\} \to \{1,2,3\}$

$f(\text{Khalil}) = 1 \quad f(\text{Shahzar}) = 1 \quad f(\text{Patrick}) = 1$

2. Write down a formula that says where each element of the domain is sent

Bad $\quad h: \mathbb{N} \to \mathbb{N} \quad h(x) = x - 1$

Bad $\quad K: \mathbb{R} \to \mathbb{R} \quad K(x) = \frac{1}{x}$

3. Describe in words where each elt of the domain is sent

$j: \{0, 1, 2, \ldots, 10\} \to \mathbb{N}$

$j(n)$ = the smallest natural number with at least n letters when written in English

$j(4) = \qquad j(5) =$

4. Definition by cases

$\ell: \mathbb{N} \to \mathbb{N}$

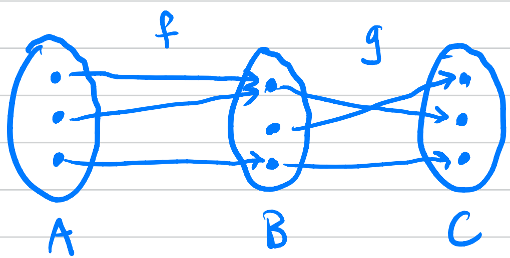
$$\ell(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n-1/2 & \text{if } n \text{ is odd} \end{cases}$$

$\ell(2) =$

$\ell(9) =$

3.3 Operations on functions

Def If $f: A \to B$ and $g: B \to C$ are functions, the composition of g and f, denoted $g \circ f$ is the function from A to C defined by



$$A \xrightarrow{f} B \xrightarrow{g} C$$

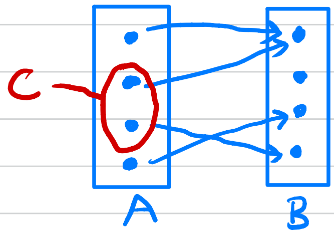
Example $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ $\qquad f(n) = (n, n+1)$

$g: \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ $\qquad g((n,m)) = \sqrt{n \cdot m}$

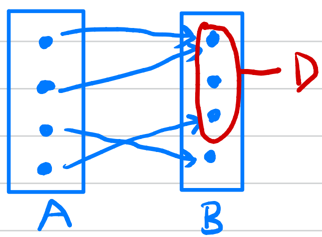
$(g \circ f):$ $\qquad (g \circ f)(n) =$

Have a function $f: A \to B$

<u>Def</u> If $C \subseteq A$, the **image of C under f**, written **$f(C)$**, is the set $\{b \in B \mid \exists x \in C \ (f(x) = b)\}$

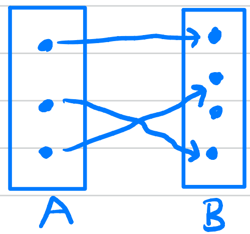


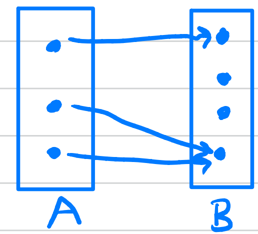
<u>Def</u> If $D \subseteq B$, the **inverse image** of D under f, written **$f^{-1}(D)$**, is the set $\{a \in A \mid f(a) \in D\}$



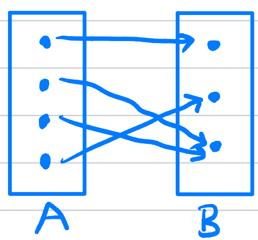
④ Special Properties of Functions

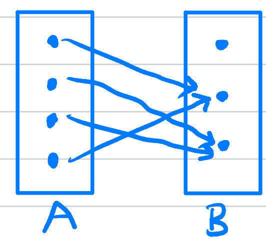
Def A function $f: A \to B$ is injective if





Def A function $f: A \to B$ is surjective if





Tip: To prove f injective, use contrapositive

Examples

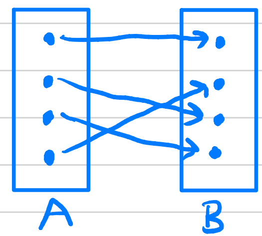
① $f: \mathbb{N} \to \mathbb{N}$
$f(n) = 2n$

② $g: \mathbb{Z} \to \mathbb{N}$
$g(n) = |n|$

③ $h: \mathbb{Z} \to \mathbb{Z}$
$h(n) = n+1$

④ $k: \{1, 2\} \to \{a, b, c\}$
$k(1) = a$
$k(2) = a$

Def A function $f: A \to B$ is bijective if



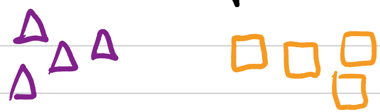
Def If $f: A \to B$ is a bijection, the inverse of f, denoted f^{-1}, is the function from B to A defined by

$f^{-1}(b) =$

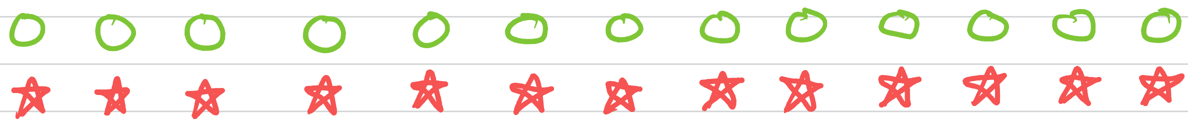
For all $a \in A$,
For all $b \in B$,

⑤ Cardinality

Are there the same number of △ and □ ?



Are there the same number of ○ and ★ ?



Did you have to count?

Idea for today:

An aside

Ask a very young child:

Are there the same number of △ and □ ?

△ △ △ △ △

□ □ □ □ □

Are there the same number of ○ and ☆ ?

○ ○ ○ ○ ○

☆ ☆ ☆ ☆ ☆

5.1 Definition of Cardinality

Def Two sets A and B have the same cardinality, written $|A|=|B|$, if

Example $|\{1,2,3\}| = |\{a,b,c\}|$

Exercise ① Show $|A|=|B| \Rightarrow |B|=|A|$

② Show $(|A|=|B| \wedge |B|=|C|) \Rightarrow |A|=|C|$

Thm Let A be the set of even natural numbers.
Then $|\mathbb{N}| = |A|$.

proof

Thm $|\mathbb{N}| = |\mathbb{Z}|$

proof

Thm $|\mathbb{N}| = |\mathbb{Q}^+|$

proof

	1	2	3	4	5	....
1						
2						
3						
4						

(1)

(2)

(3)

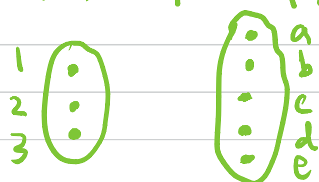
Exercise (1)

(2)

(5.2) More about cardinality

Def If A and B are sets, we say the cardinality of A is less than or equal to the cardinality of B, written $|A| \le |B|$, if

Example $|\{1,2,3\}| \le |\{a,b,c,d,e\}|$



Note: $|A| = |B| \Rightarrow |A| \le |B|$

Thm (Cantor-Schroeder-Bernstein) If $|A| \le |B|$ and $|B| \le |A|$ then

Meaning:

6 Countability

Def A set A is countable if

Some countable sets

A set is countable iff it is either

A is countable =

⑦ Uncountability

Def A set is uncountable if

Are there any uncountable sets?

Thm (Cantor) (will prove on next slide)

Cor

Note: Bijection $f: \mathbb{N} \to \mathbb{R}$ =

Thm (Cantor) $|\mathbb{N}| \neq |\mathbb{R}|$

proof