0 Some advice on solving problems

Maxim # 1 Don't give up after 30 seconds!

Common pattern: look at a problem, can't think of anything right away, give up.

Instead: Think about the problem for 5–10 minutes. often you will at least come up with some things to try.

Maxim #2 Do the messy calulations/hard work!

Common pattern: you have an idea for how to solve a problem but it looks hard to do & maybe won't work anyway, so you look for an easier approach

Instead: Just try it! 30% of the time it will work

① Introduction

<u>Question</u> Are there more real numbers than natural numbers?

?? There are infinitely many of both!
What does "more" mean?

<u>Goal for today</u> Turn this into a mathematical question and answer it.

Key step: finding a reasonable mathematical definition of when two infinite sets have "the same number" of elements

Not so important for CS. But an excuse to introduce useful mathematical terminology. Plus it's super cool!

② Sets

Def A set is an unordered collection of objects

Two sets are equal if they contain the same objects

Example $\{1,2,3\} = \{3,1,2\}$  ← bag

$\{2,3,5\}$ = set of all prime numbers < 6

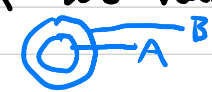
$\{1,2,3\} \neq \{1,2,3,4\}$

Def $x \in A$ means x is an element of A

$x \notin A$ means x is not an element of A

Example $2 \in \{1,2,3\}$, $0 \notin \{1,2,3\}$, $\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$

Def A set A is a subset of B, written $A \subseteq B$, if for every $x \in A$ we have $x \in B$ $\quad \forall x\,(x \in A \Rightarrow x \in B)$



Example $\{1,2,3\} \subseteq \{1,2,3,4\}$, $\mathbb{N} \subseteq \mathbb{R}$, $\{1,2\} \subseteq \{1,2\}$, $\{-1\} \not\subseteq \mathbb{N}$

Question Suppose $A \subseteq B$ and $B \subseteq A$. Must $A = B$? Yes.

How to prove two sets are equal: $\forall x \, (x \in A \Leftrightarrow x \in B)$

① Let x be an arbitrary element of A.
Show $x \in B$.

② Let x be an arbitrary element of B.
Show $x \in A$.

2.1 Some common sets

$\mathbb{N}$	natural numbers	$\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	rational numbers	$\frac{1}{2} \in \mathbb{Q}$, $\pi \notin \mathbb{Q}$
$\mathbb{R}$	real numbers	$\sqrt{2} \in \mathbb{R}$, $\pi \in \mathbb{R}$
$\emptyset$	empty set	nothing is an element of $\emptyset$

Question

1. $3 \in \mathbb{R}$? Yes
2. $3 \in \emptyset$? No!
3. $\mathbb{N} \in \mathbb{R}$? No

   not a real number!
4. $\emptyset \subseteq \mathbb{N}$? Yes (vacuously)
5. $\mathbb{N} \subseteq \emptyset$? No!
6. Steph Curry $\in \emptyset$? No!

2.2 How to describe a set

1. List the elements $\{18, 26, -3\}$, $\{\text{Khalil, Shahzar, Patrick}\}$

← curly brackets denote set (instead of list)

2. List enough elements to make it clear what set it is

$\{0, 2, 4, 6, \ldots\}$ even natural numbers

$\{1, 2, 3, \ldots, 10\}$ natural numbers between 1 and 10

Bad: $\{1, \ldots\}$ all numbers ≥ 1? all odd numbers?

3. Describe it in words "the set of all odd natural numbers"

4. Set builder notation $\{x \in A \mid P(x)\}$

(a.k.a. set comprehension)

↖ can just write $\{x \mid P(x)\}$

$$\{n \in \mathbb{N} \mid \exists m \in \mathbb{N}\,(n = 2 \cdot m)\} = \{0, 2, 4, 6, \ldots\}$$

$$\{x \in \mathbb{R} \mid x^2 = -x\} = \{0, -1\}$$

$$\{x \in \mathbb{N} \mid x^2 = -x\} = \{0\}$$

5. Some specialized notation. $a \leq b$ real numbers

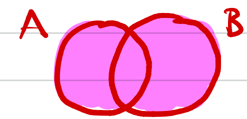
$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

2.3 Operations on sets

① **Union** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$\{1,2\} \cup \{1,7\} = \{1,2,7\}$

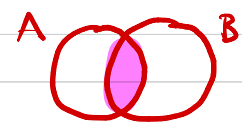
$\{1,2,3\} \cup \emptyset = \{1,2,3\}$



② **Intersection** $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

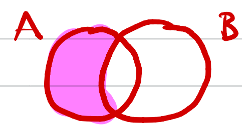
$\{1,2\} \cap \{1,7\} = \{1\}$

$\{1,2\} \cap \{4,3\} = \emptyset$ ← disjoint



③ **Difference** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

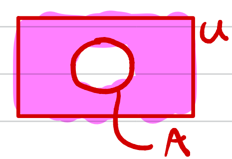
$\{1,2\} \setminus \{1,7\} = \{2\}$



④ **Complement** Assume $A \subseteq U$. $A^c = U \setminus A$ → sometimes $\bar{A}$

irrational numbers $= \mathbb{Q}^c$

→ $\mathbb{R}$ is implicit



ordered pair

⑤ Cartesian product $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

$\{1,2\} \times \{2,3\} = \{(1,2),(1,3),(2,2),(2,3)\}$

Also $A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1 \wedge \dots \wedge a_n \in A_n\}$

<u>Example</u> $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$ ← EE16A/Math 54

⑥ Powerset $\mathcal{P}(A)$ = set of all subsets of A

$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

<u>Question</u> $\emptyset \in \mathcal{P}(A)$ for any A? Yes. $\emptyset \subseteq A$

$\mathbb{N} \in \mathcal{P}(\mathbb{R})$? Yes. $\mathbb{N} \subseteq \mathbb{R}$

$A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$? Yes $X \subseteq A \Rightarrow X \subseteq B$

2.4 Cardinality of finite sets

Def If A is a finite set, write $|A|$ to denote the number of elements in A

$|\{-1, 7\}| = 2$

$|\{\text{Khalil}, \text{Shahzar}, \text{Patrick}\}| = 3$

Question If $|A| = |B| = 2$, what is $|A \times B|$? 4

$A = \{a_1, a_2\}$ $B = \{b_1, b_2\}$ ← all distinct

$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$

Exercise If $|A| = n$ and $|B| = k$ then $|A \times B| = nk$

If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$

③ Functions

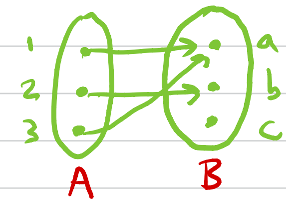
<u>Def</u> If A and B are sets, a **function** f from A to B is an assignment of exactly one element of B to each element of A

domain codomain

$f: A \to B$

<u>Example</u> $f: \{1,2,3\} \longrightarrow \{a,b,c\}$

$f(1) = a \quad f(2) = b \quad f(3) = a$



$g: \mathbb{R} \to \mathbb{R}$
$g(x) = x^2$

<u>Def</u> The **range** of a function $f: A \to B$, written **range(f)**, is the set $\{ b \in B \mid \exists a \in A \ (f(a) = b) \}$

<u>Example</u> With f and g as above,

range$(f) = \{a, b\}$

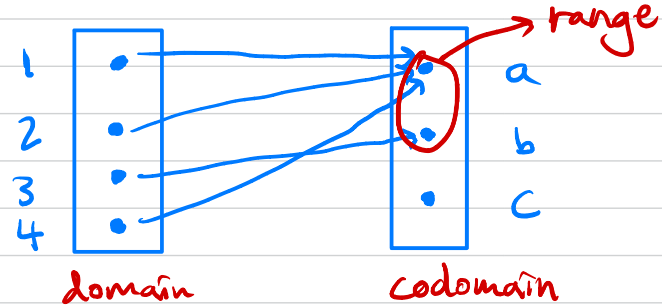
range(g) = set of all real numbers ≥ 0.

3.1 Anatomy of a function

domain

codomain (part of the def. of f)

$$f : A \longrightarrow B$$



$f(1) = a$

image of 1 under f

3.2 How to describe functions

1. Explicitly list where each element of the domain is sent

$f: \{\text{Khalil}, \text{Shahzar}, \text{Patrick}\} \to \{1,2,3\}$

$f(\text{Khalil}) = 1 \quad f(\text{Shahzar}) = 1 \quad f(\text{Patrick}) = 1$

2. Write down a formula that says where each element of the domain is sent

$g: \mathbb{N} \to \mathbb{N} \qquad g(x) = x+1$

Bad $\quad h: \mathbb{N} \to \mathbb{N} \qquad h(x) = x-1 \qquad h(0) \notin \mathbb{N} \qquad \mathbb{N} \to \mathbb{Z}\,?$

Bad $\quad k: \mathbb{R} \to \mathbb{R} \qquad k(x) = \frac{1}{x} \qquad k(0) = ?? \qquad \mathbb{R}\setminus\{0\} \to \mathbb{R}\,?$

3. Describe in words where each elt of the domain is sent

$j: \{0,1,2,\ldots,10\} \to \mathbb{N}$

$j(n) =$ the smallest natural number with at least n letters when written in English

$j(4) = 0 \qquad j(5) = 3$

4. Definition by cases

$\ell: \mathbb{N} \to \mathbb{N}$

$$\ell(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n-1/2 & \text{if } n \text{ is odd} \end{cases}$$

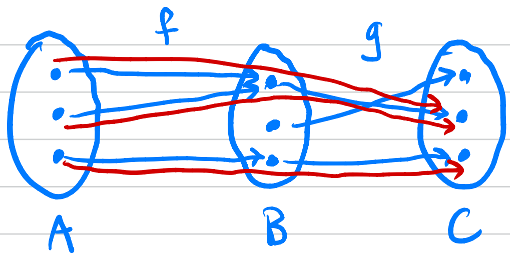
$\ell(2) = 1$

$\ell(9) = 4$

3.3 Operations on functions

Def If $f: A \to B$ and $g: B \to C$ are functions, the composition of g and f, denoted $g \circ f$ is the function from A to C defined by

$$(g \circ f)(a) = g(f(a))$$ "first do f then do g"



$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

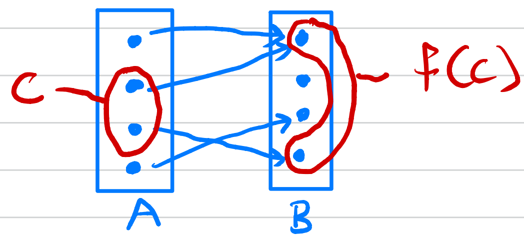
Example $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ $\quad f(n) = (n, n+1)$

$g: \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ $\quad g((n,m)) = \sqrt{n \cdot m}$

$(g \circ f): \mathbb{N} \to \mathbb{R}$ $\quad (g \circ f)(n) = g((n, n+1)) = \sqrt{n(n+1)}$

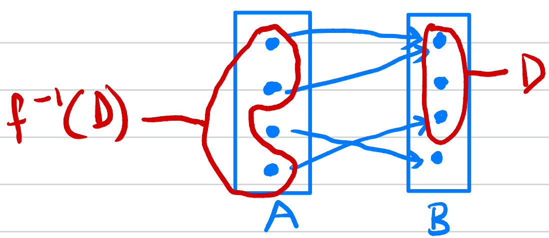
Have a function $f: A \to B$

<u>Def</u> If $C \subseteq A$, the image of C under f, written $f(C)$, is the set $\{b \in B \mid \exists x \in C \, (f(x) = b)\}$



$f(C) \subseteq B$

<u>Def</u> If $D \subseteq B$, the inverse image of D under f, written $f^{-1}(D)$, is the set $\{a \in A \mid f(a) \in D\}$



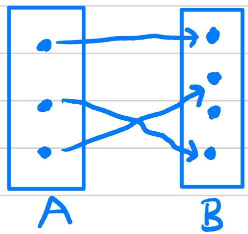
"things that get sent into D"

$f^{-1}(D) \subseteq A$

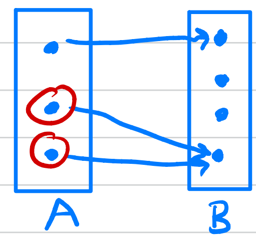
④ Special Properties of Functions

= one-to-one

Def A function $f: A \to B$ is injective if for all $a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$



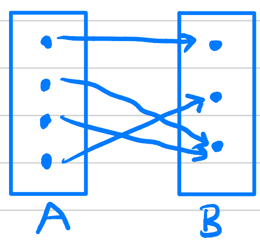
injective



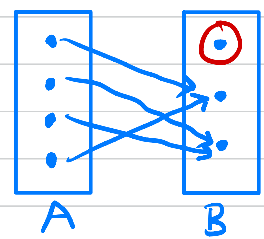
not injective

= onto

Def A function $f: A \to B$ is surjective if for all $b \in B$ there is some $a \in A$ such that $f(a) = b$ $\equiv$ $\text{range}(f) = B$



surjective



not surjective

Tip: To prove f injective, use contrapositive

$$\forall a_1, a_2 \in A \ (a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2))$$
$$\equiv \ \forall a_1, a_2 \in A \ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2)$$

Examples

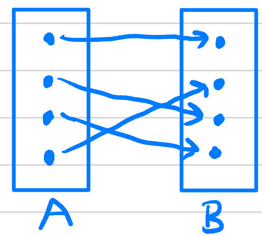
① $f: \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$ — injective: $2n = 2m \Rightarrow n = m$; not surjective: $1 \notin \text{range}(f)$

② $g: \mathbb{Z} \to \mathbb{N}$, $g(n) = |n|$ — not injective: $|-2| = |2|$; surjective: if $n \in \mathbb{N}$ then $g(n) = n$

③ $h: \mathbb{Z} \to \mathbb{Z}$, $h(n) = n+1$ — injective: $n+1 = m+1 \Rightarrow n = m$; surjective: $\forall n,\ h(n-1) = n$

④ $k: \{1, 2\} \to \{a, b, c\}$, $k(1) = a$, $k(2) = a$ — not injective: $k(1) = k(2)$; not surjective: $b \notin \text{range}(k)$

<u>**Def**</u> A function $f: A \to B$ is **bijective** if it is both injective and surjective. f is called a bijection



bijective

<u>**Def**</u> If $f: A \to B$ is a bijection, the **inverse** of f, denoted f^{-1}, is the function from B to A defined by

$$f^{-1}(b) = \text{the unique } a \text{ such that } f(a) = b$$

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

For all $a \in A$, $f^{-1}(f(a)) = a$

For all $b \in B$, $f(f^{-1}(b)) = b$

⑤ Cardinality

Are there the same number of △ and □ ?

△ △ △ △ □ □ □ □ Yes! 4 of each

Are there the same number of ○ and ☆ ?

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆ ☆

Yes!

Did you have to count? No!

Can match up ○ and ☆ so that all are paired up

I.e. there is a bijection between the two sets!

Idea for today: Extend this definition to infinite sets!

An aside

Ask a very young child:

Are there the same number of △ and □ ?

△ △ △ △ △
□ □ □ □ □

they will say "yes"

Are there the same number of ○ and ☆ ?

○ ○ ○ ○ ○
☆ ☆ ☆ ☆ ☆

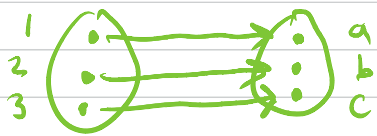
they will say "no"

Counting is not that obvious! But bijections are?

5.1 Definition of Cardinality

Def Two sets A and B have the same cardinality, written $|A| = |B|$, if there is a bijection $f: A \to B$

Example $|\{1,2,3\}| = |\{a,b,c\}|$ $f: \{1,2,3\} = \{a,b,c\}$
$f(1) = a$ $f(2) = b$ $f(3) = c$

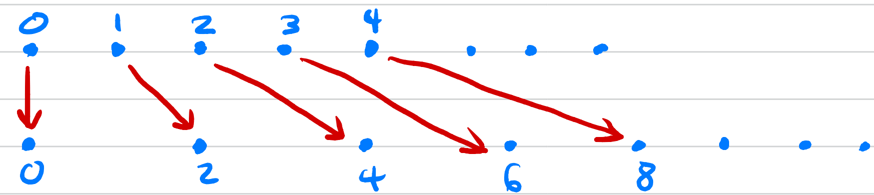


Exercise ① Show $|A| = |B| \Rightarrow |B| = |A|$
Hint: inverse of a bijection

② Show $(|A| = |B| \land |B| = |C|) \Rightarrow |A| = |C|$
Hint: composition of functions

Thm Let A be the set of even natural numbers.
Then $|\mathbb{N}| = |A|$.

proof



$f : \mathbb{N} \to A$ defined by $f(n) = 2n$
is a bijection

Thm $|\mathbb{N}| = |\mathbb{Z}|$

proof

$$
\begin{aligned}
0 &\longmapsto 0 \\
1 &\longmapsto 1 \\
2 &\longmapsto -1 \\
3 &\longmapsto 2 \\
4 &\longmapsto -2 \\
&\vdots
\end{aligned}
$$

$f : \mathbb{N} \to \mathbb{Z}$ defined by

$$
f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}
$$

Thm $|\mathbb{N}| = |\mathbb{Q}^+|$ ← positive rational numbers

proof

	1	2	3	4	5	...
1	1/1	1/2	1/3	1/4	1/5	...
2	2/1	2/2	2/3	2/4	2/5	...
3	3/1	3/2	3/3	...		
4	4/1	4/2	...			
⋮	⋮					

① Make infinite grid with $\mathbb{N}^+$ on each axis

② Follow zig-zag path

③ Cross out duplicates

$$
\begin{aligned}
0 &\longmapsto 1/1 = 1\\
1 &\longmapsto 1/2\\
2 &\longmapsto 2/1 = 2\\
3 &\longmapsto 3/1 = 3\\
4 &\longmapsto 1/3\\
5 &\longmapsto 1/4\\
&\vdots
\end{aligned}
$$

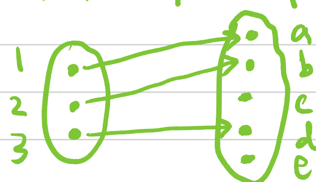
Exercise ① $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

② $|\mathbb{N}| = |\mathbb{Q}|$

(5.2) More about cardinality

<u>Def</u> If A and B are sets, we say the cardinality of A is less than or equal to the cardinality of B, written $|A| \le |B|$, if there is an injective function $f: A \to B$

<u>Example</u> $|\{1,2,3\}| \le |\{a,b,c,d,e\}|$



Note: $|A| = |B| \Rightarrow |A| \le |B|$ A bijection <u>is</u> an injection

<u>Thm</u> (Cantor-Schroeder-Bernstein) If $|A| \le |B|$ and $|B| \le |A|$ then $|A| = |B|$. → Not that easy!

Meaning: to show $|A| = |B|$, enough to find injections going both ways

⑥ Countability

Def A set A is countable if $|A| \leq |\mathbb{N}|$.

Some countable sets

- Any finite set
- Set of even natural numbers
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}^+$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}$
- Set of all finite binary sequences $\{0, 1, 00, 01, 10, 11, 000, \ldots\}$
- Set of all valid Java programs

A set is countable iff it is either finite or has the same cardinality as $\mathbb{N}$

A is countable = you can list the elements of A

i.e. $A = \{a_1, a_2, a_3, \ldots\}$

7 Uncountability

Def A set is uncountable if it is not countable.

Are there any uncountable sets? Yes!

Thm (Cantor) $|\mathbb{N}| \neq |\mathbb{R}|$ (will prove on next slide)

Cor $\mathbb{R}$ is not countable

$|\mathbb{N}| \leq |\mathbb{R}|$ $\quad f: \mathbb{N} \to \mathbb{R}$ $\quad f(n) = n$

So if $|\mathbb{R}| \leq |\mathbb{N}|$ then by CSB, $|\mathbb{N}| = |\mathbb{R}|$, contradicting Cantor's theorem

Note: Bijection $f: \mathbb{N} \to \mathbb{R}$ = way to list all real numbers

$r_0 = f(0),\ r_1 = f(1),\ r_2 = f(2), \ldots$

$\mathbb{R} = \{r_0, r_1, r_2, \ldots\}$

Thm (Cantor) $|\mathbb{N}| \neq |\mathbb{R}|$

proof Suppose we can list all real numbers:

$$\mathbb{R} = \{r_0, r_1, r_2, r_3, \ldots\}$$

We will show there is a real number not on this list!

Write r_i's in decimal notation

E.g.

$$r_0 = -17.0287651\ldots$$
$$r_1 = 8.111137\ldots$$
$$r_2 = 0.00000000\ldots$$
$$r_3 = 6.7159328\ldots$$
$$\vdots$$

$r = 0.1011\ldots$

Comment: need to be a little careful

$0.9999\ldots = 1.0000\ldots$

Define r so that its n^{th} digit after the decimal point is not equal to the n^{th} digit of r_n

$$r = 0.d_0 d_1 d_2 \ldots \quad \text{where} \quad d_n = \begin{cases} 0 & \text{if } n^{th} \text{ digit of } r_n \text{ is } 1 \\ 1 & \text{otherwise} \end{cases}$$

r disagrees with every r_n, so it is not on our list!