Definitions

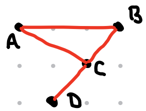
<u>Def</u> An undirected graph $G=(V,E)$ is defined by

① a set V of

② a set E of

where elements in E

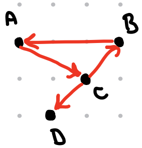
<u>Ex</u> ①



②

③

<u>Note</u> To make an directed graph $G=(V,E)$, we can define

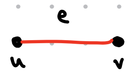


<u>Def</u> Given an edge $e=\{u,v\}$, we say

e is to u and v

u and v are

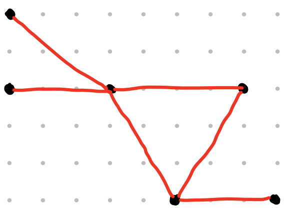
u and v are



The degree of a vertex v is

$\deg(v)=$

<u>Ex</u>

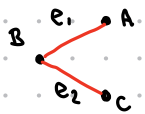


Handshaking Lemma

<u>Thm</u> (Handshaking Lemma) Let $G=(V,E)$ be a graph with $m=|E|$ edges. Then

$$2m = \sum_{v\in V} \deg(v)$$

<u>Ex</u> $m=2$



<u>Pf</u> Let N be the number of vertex edge pairs (v,e) such that v is incident to e.

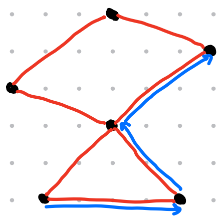
Therefore

Walking on Graphs

Def A walk is
A walk is closed if
A walk is open if

A path is
A tour is
A cycle is
An Eulerian tour is

Ex ①



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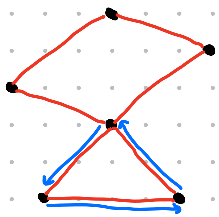
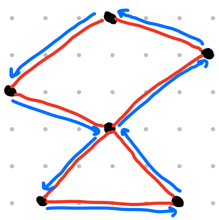


③

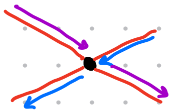


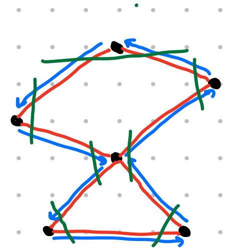
Def A graph is connected if

Eulerian Tours

<u>Thm</u> A connected graph $G=(V,E)$ has an Eulerian tour iff every vertex has even degree

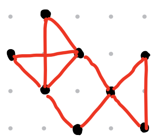
<u>Pf</u> $\Rightarrow$) Suppose G has an Eulerian tour starting at some vertex v_0.





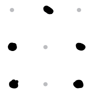
$\Leftarrow$) Suppose every vertex in G has even degree

<u>Ex</u> ★ Use the above algorithm to find an Eulerian tour in the following graph.



Graph Families

Def A complete graph on n vertices, denoted , is a graph with n vertices and all possible edges.
Ex

K_5 

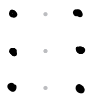
Note For K_n,

$|E| =$

Def A bipartite graph partitions its vertex set V into two disjoint sets L and R such that

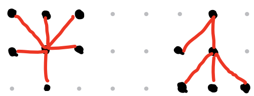
$E \subseteq$

A complete bipartite graph, denoted , has $|L| =$, $|R| =$, and $E = \{\{u,v\} : u \in L, v \in R\}$.

$K_{3,3}$ 

Note For $K_{n,m}$,
$|E| =$

Def A tree is a
Ex

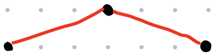


Thm The following statements about a graph $T = (V, E)$ are equivalent.
- T is connected and acyclic
- T is connected and has $|V| - 1$ edges
- T is connected and removing any edge disconnects T
- T has no cycles and adding any edge creates a cycle.

Planar Graphs

Def A graph is called   if it can be drawn in the plane without any edges crossing.
Such a drawing is called a planar representation.

Ex ①



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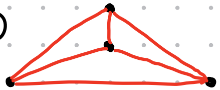


③

Thm (Euler's Formula) Let G be a connected planar graph.

v = # vertices

e = # edges

f = # faces (a face is a region bounded by edges in the planar representation)

Then

$$v - e + f = 2$$

Pf By induction on

Base case:

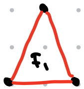
Inductive hypothesis: Suppose that for any connected planar graph

Inductive step:

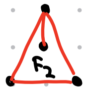
Sparsity

Cor For a connected planar graph with $v \geq 3$, we have $e \leq 3v - 6$

Pf Define the degree of a face to be the # of edges on its boundary, where edges are counted twice if they have the face on both sides.



$\deg(F_1) =$



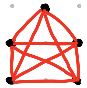
$\deg(F_2) =$

Cor For a connected planar bipartite graph with $v \geq 3$, $e \leq 2v - 4$

Pf

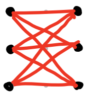
Nonplanarity

Ex K_5



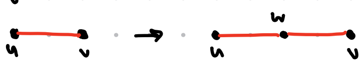
Pf Suppose for contradiction K_5 is planar.

Ex $K_{3,3}$



Pf Suppose for contradiction $K_{3,3}$ is planar.

Def An elementary subdivision of $G=(V,E)$ is an operation which replaces an edge $\{u,v\} \in E$ with new edges $\{u,w\}$ and $\{w,v\}$.



Def: Two graphs G_1 and G_2 are homeomorphic if they can be obtained from the same graph by sequences of elementary subdivisions.

Thm (Kuratowski) A graph is nonplanar iff it contains a subgraph homeomorphic to $K_{3,3}$ or K_5