Definitions

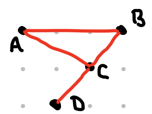
Def An undirected graph $G=(V,E)$ is defined by

① a set V of vertices

② a set E of edges

where elements in E are of the form $\{u,v\}$ for $u,v \in V$ and $u \neq v$.

Ex ①



This is a graph!

$V = \{A, B, C, D\}$

$E = \{\{A,B\}, \{A,C\}, \{B,C\}, \{C,D\}\}$

②

Not a (simple) graph

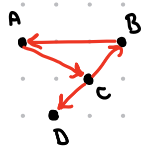
$E = \{\{A,B\}, \{A,B\}\}$ is not a set

③

Not a (simple graph)

$E = \{\{A,A\}\}$ is not a set

Note To make an directed graph $G=(V,E)$, we can define $E \subseteq V \times V$.

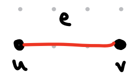


$V = \{A, B, C, D\}$

$E = \{(A,C), (B,A), (C,B), (C,D)\}$

Def Given an edge $e = \{u,v\}$, we say

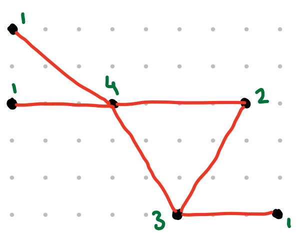
- e is incident to u and v
- u and v are neighbors
- u and v are adjacent



The degree of a vertex v is the number of incident edges.

$\deg(v) = |\{v \in V : \{u,v\} \in E\}|$

Ex

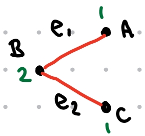


Handshaking Lemma

Thm (Handshaking Lemma) Let $G=(V,E)$ be a graph with $m=|E|$ edges. Then

$$2m = \sum_{v\in V} \deg(v)$$

Ex $m=2$



$2(2) = 4 = 1+2+1$

Observation: we can associate each edge to two vertices.

$$\overbrace{(A,e_1)}^{\deg A}\ \overbrace{(B,e_1),\ (B,e_2)}^{\deg B}\ \overbrace{(C,e_2)}^{\deg C}$$

$$\underbrace{(A,e_1)\ (B,e_1)}_{e_1},\ \underbrace{(B,e_2)\ (C,e_2)}_{e_2}$$

Pf Let N be the number of vertex edge pairs (v,e) such that v is incident to e.

- Each vertex v is incident to $\deg(v)$ edges, so

$$\sum_{v\in V} \deg(v) = N$$

- Each edge is incident to two vertices, so

$$2m = N$$

Therefore

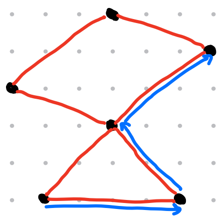
$$2m = N = \sum_{v\in V} \deg(v) \qquad \square$$

Walking on Graphs

Def A walk is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$.
A walk is closed if the start and end vertices are the same $(v_1 = v_n)$
A walk is open if the start and end vertices are different $(v_1 \neq v_n)$

A path is an open walk with no repeated vertices ($\Rightarrow$ no repeated edges)
A tour is a closed walk with no repeated edges
A cycle is a tour with no repeated vertices other than v_1 and v_n
An Eulerian tour is a tour which visits every edge exactly once

Ex ①



path

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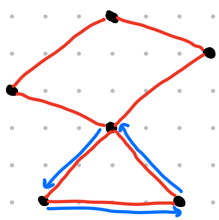
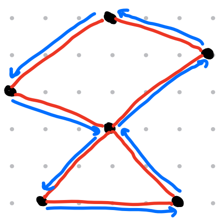


cycle

③



tour (Eulerian)

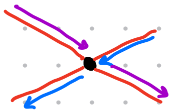
Def A graph is connected if there exists a path between any two distinct $u, v \in V$.

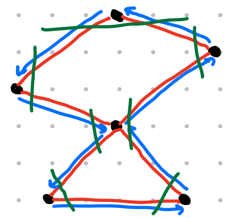
Eulerian Tours

Thm A connected graph $G=(V,E)$ has an Eulerian tour iff every vertex has even degree

Pf $\Rightarrow$) Suppose G has an Eulerian tour starting at some vertex v_0.

For all $v \in V$, pair up two edges each time the vertex is traversed.





For v_0, pair up the starting and ending edges.

The Eulerian tour visits each edge exactly once

$\Rightarrow \forall v \in V$, all incident edges are paired

$\Rightarrow \forall v \in V$, $\deg(v)$ is even

$\Leftarrow$) Suppose every vertex in G has even degree

We must construct an Eulerian tour. Use the following algorithm.

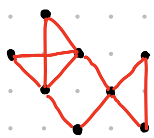
1. Pick arbitrary $v_0 \in V$

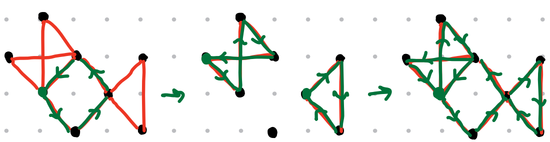
   Visit unvisited edges until you no longer can

   Since all degrees are even, we'll get stuck at v_0
2. Remove the tour created by 1

   Recurse on the connected components
3. Splice recursive tours together

Ex ★ Use the above algorithm to find an Eulerian tour in the following graph.

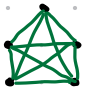




Graph Families

Def A complete graph on n vertices, denoted K_n, is a graph with n vertices and all possible edges.

Ex

K_5 

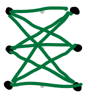
Note For K_n,

$$|E| = \frac{n(n-1)}{2}$$

Def A bipartite graph partitions its vertex set V into two disjoint sets L and R such that

$$E \subseteq \{\{u,v\} : u \in L,\ v \in R\}$$

A complete bipartite graph, denoted $K_{n,m}$, has $|L| = n$, $|R| = m$, and $E = \{\{u,v\} : u \in L,\ v \in R\}$.

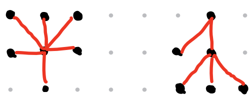
$K_{3,3}$ 

Note For $K_{n,m}$,

$|E| = nm$

Def A tree is a connected, acyclic graph

Ex



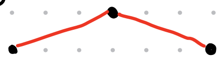
Thm The following statements about a graph $T = (V, E)$ are equivalent.

- T is connected and acyclic
- T is connected and has $|V| - 1$ edges
- T is connected and removing any edge disconnects T
- T has no cycles and adding any edge creates a cycle.

Planar Graphs

Def A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation.

Ex ①



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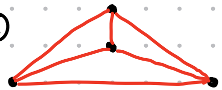

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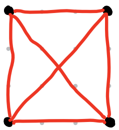
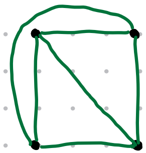
 $\equiv$ 

Thm (Euler's Formula) Let G be a connected planar graph.

v = # vertices
e = # edges
f = # faces (a face is a region bounded by edges in the planar representation)

Then

$$v - e + f = 2$$

Pf By induction on e.

Base case: $e = 0$

• $v = 1$, $f = 1$
$1 - 0 + 1 = 2$

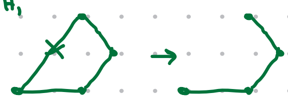
Inductive hypothesis: Suppose that for any connected planar graph with k edges, $v - e + f = 2$

Inductive step: Consider any connected planar graph G with $k+1$ edges. There are two cases.

① If G is a tree, then $v = (k+1)+1$, $f = 1$. So $v - e + f = (k+1)+1 - (k+1) + 1 = 2$.

② If G is not a tree, it must have a cycle. Remove any edge from any cycle to yield a connected planar graph with v vertices, k edges, and $f-1$ faces. By the IH,

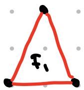
$v - k + (f-1) = 2$
$v - (k+1) + f = 2$ $\square$

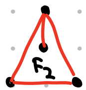


Sparsity

Cor For a connected planar graph with $v \geq 3$, we have $e \leq 3v-6$

Pf Define the degree of a face to be the # of edges on its boundary, where edges are counted twice if they have the face on both sides.





$\deg(f_1) = 3$ $\deg(f_2) = 5$

Then

$$\sum_{i=1}^{f} \deg(f_i) = 2e$$

since each edge is incident to two faces.

Since $\deg(f_i) \geq 3$ for any i,

$$2e = \sum_{i=1}^{f} \deg(f_i)$$

$$\geq \sum_{i=1}^{f} 3 = 3f$$

So $2e \geq 3f$, or, equivalently, $f \leq \frac{2}{3}e$

By planarity,

$v - e + f = 2$

$f = 2 + e - v$, $f \leq \frac{2}{3}e \Rightarrow 2 + e - v \leq \frac{2}{3}e$

$2 - v \leq -\frac{1}{3}e$

$e \leq 3v - 6$ $\square$

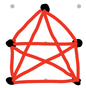
Cor For a connected planar bipartite graph with $v \geq 3$, $e \leq 2v-4$

Pf First, prove that a bipartite graph has no odd-length cycles.

Therefore $\deg(f_i) \geq 4$. Finish from there.

Nonplanarity

Ex K_5



Pf Suppose for contradiction K_5 is planar.

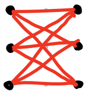
Then $e \leq 3v-6$.

$e = \frac{5(5-1)}{2} = 10$ $\quad v = 5$

$10 \leq 3(5)-6 = 11$

Contradiction. $\square$

Ex $K_{3,3}$



Pf Suppose for contradiction $K_{3,3}$ is planar.

Then $e \leq 3v-6$

$e = 3(3) = 9$ $\quad v = 6$

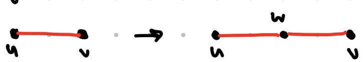
$9 \leq 3(6)-6 = 12$

No contradiction... Since $K_{3,3}$ bipartite, $e \leq 2v-4$

$9 \leq 2(6)-4 = 8$

Contradiction $\square$

Def An elementary subdivision of $G=(V,E)$ is an operation which replaces an edge $\{u,v\} \in E$ with new edges $\{u,w\}$ and $\{w,v\}$.



Def: Two graphs G_1 and G_2 are homeomorphic if they can be obtained from the same graph by sequences of elementary subdivisions.

Thm (Kuratowski) A graph is nonplanar iff it contains a subgraph homeomorphic to $K_{3,3}$ or K_5