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① Why Counting?

Preview: If you have a finite set S of equally likely outcomes, the probability of an event A is given by:

$$P(A) = \frac{|A|}{|S|}$$

Example: Fair coin flip

A = coin lands heads

$$A = \{H\}$$

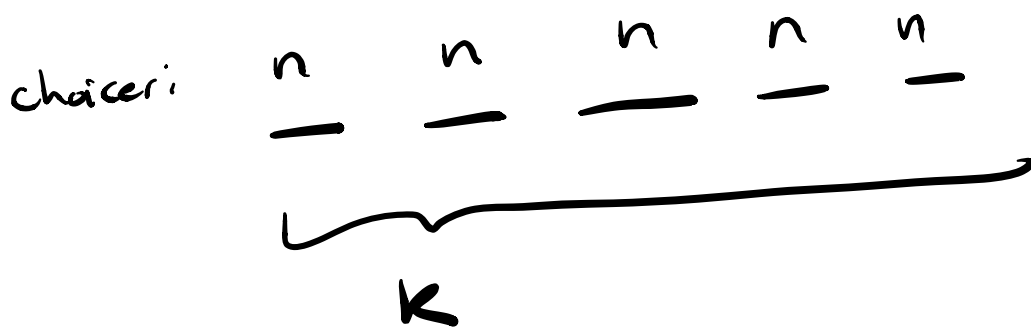
$$S = \{H, T\}$$

$$\rightarrow P(A) = \frac{|A|}{|S|} = \frac{1}{2}$$

Need to be able to count the number of elements in a set in an efficient manner

② Ordered Sampling with Replacement

Consider a set S with n elements, we want to draw k such that order matters and repetition is allowed.



→ total number of ways is n^k

In general, First Rule of Counting:

If k choices in succession, where n_1 options for first choice, and for each first choice you have n_2 options for second choice, and so on, then the total number of ways to make the k choices is

$$n_1 \times n_2 \times \dots \times n_k.$$

Example

$$S = \{1, 2, 3\}$$

How many ways
with replacement

to choose 2 elements,
if order matters?

③ Ordered Sampling Without Replacement.

Consider the same setting as in (2), but without replacement.

Using first rule of counting,

$$\text{choices } \underbrace{\underline{n} \quad \underline{n-1} \quad \underline{n-2} \quad \dots \quad \underline{n-k+1}}_k$$

\Rightarrow Total number of ways is

$$n \cdot (n-1) \cdot \dots \cdot (n-k+1) = {}_n P_k$$

" n permute k "

$${}_n P_k = \frac{n!}{(n-k)!}$$

Example

$$S = \{1, 2, 3\}$$

How many ways are there to choose 2 elements without replacement & order matters?

④ Unordered sampling without replacement

Consider a set S with n elements, we want to draw k such that order does not matter, and no replacement.

Second Rule of Counting

$$\# \text{ of ways to choose when order doesn't matter} = \frac{\# \text{ of ways to choose with order mattering}}{\# \text{ of ordered ways per unordered way.}}$$

So, $\frac{n!}{(n-k)!}$ gives k elements but order matters

There are $k!$ ways to rearrange the k elements.

So, unordered sampling w/out replacement

\Rightarrow total number of ways

$$\text{is } \frac{\frac{n!}{(n-k)!}}{k!}$$

Note: $\binom{n}{k} = \frac{n!}{(n-k)! (k)!}$
"n choose k"

Example

$$S = \{1, 2, 3\}$$

How many ways are there to choose 2 elements without replacement & order does not matter?

(5) Unordered sampling with replacement.

Consider a set S with n elements, we want to draw k such that order does not matter and repetition is allowed.

Can try to use second rule of counting but run into issue of nonconstant number of ordered ways per unordered way.

Example:

$$S = \{1, 2, 3\}$$

1 → apples

2 → bananas

3 → oranges

unlimited supply

how many ways to select 5 pieces of fruit?

Zeroth rule of Counting

Want to count size of A ?

find bijection between A & B

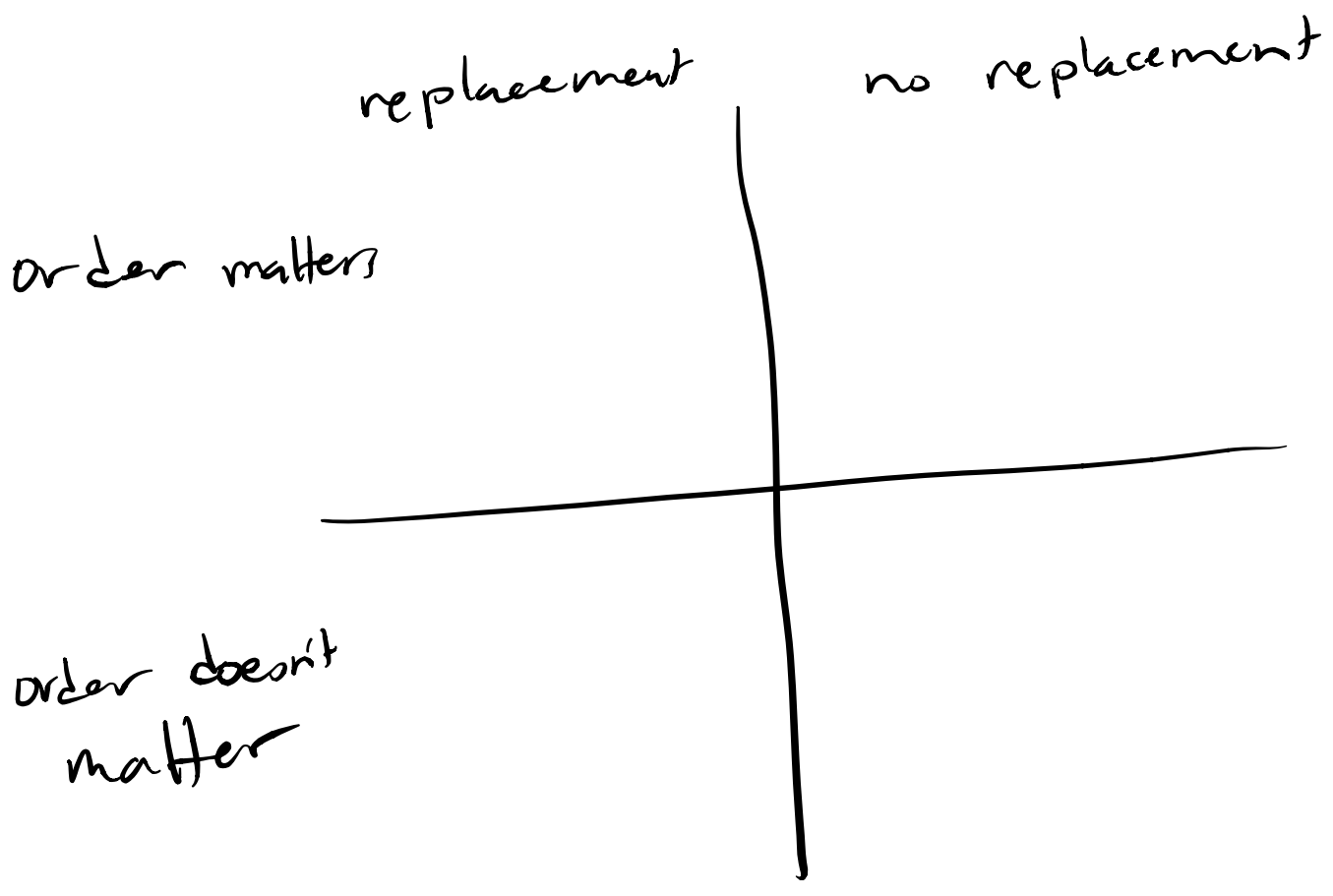
where B is easier to count, and

just count size of B .

this works since $|A| = |B|$

(6) Summary of Approaches

Number of ways to choose k items from set of size n



7 Combinatorial Proofs

Idea: Use intuitive counting arguments instead of tedious algebraic manipulation to prove some identity.

Example

Prove
$$\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \dots + \binom{k}{k}$$

Proof:

Example

Prove $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

Proof: