1. Why Counting?
2. Ordered sampling with replacement
3. Ordered sampling without replacement
4. Unordered sampling without replacement
5. Unordered sampling with replacement.
6. Summary of approaches.
7. Combinatorial Proofs

① Why Counting?

Preview: If you have a finite set S of equally likely outcomes, the probability of an event A is given by:

$$P(A) = \frac{|A|}{|S|}$$

Example: Fair coin flip

A = coin lands heads

$A = \{H\}$

$S = \{H, T\}$

$\rightarrow P(A) = \frac{|A|}{|S|} = \frac{1}{2}$

Need to be able to count the number of elements in a set in an efficient manner

(2) Ordered Sampling with Replacement

Consider a set S with n elements, we want to draw k such that order matters and repitition is allowed.

choices: $\underbrace{\underline{n}\quad \underline{n}\quad \underline{n}\quad \underline{n}\quad \underline{n}}_{k}$

→ total number of ways is $\underline{n^k}$

In general, First Rule of Counting:

If k choices in succession, where n_1 options for first choice, and for each first choice you have n_2 options for second choice, and so on, then the total number of ways to make the k choices is

$$n_1 \times n_2 \times \dots \times n_k.$$

Example

$S = \{1, 2, 3\}$

How many ways to choose 2 elements, with replacement & order matters?

$\underline{1}\ \underline{1}$

$\underline{1}\ \underline{2}$

$\underline{1}\ \underline{3}$

$\underline{2}\ \underline{1}$

$\underline{2}\ \underline{2}$

$\underline{2}\ \underline{3}$

$\underline{3}\ \underline{1}$

$\underline{3}\ \underline{2}$

$\underline{3}\ \underline{3}$

$\Rightarrow 3 \cdot 3 = 9$ ways

$n = 3$

$k = 2$

③ Ordered Sampling Without Replacement.

Consider the same setting as in ②, but without replacement.

Using first rule of counting, $n-(k-1)$

$n-0$

choices $\underline{n}$ $\underline{n-1}$ $\underline{n-2}$ $\underline{\quad}$ . . . $\underline{n-k+1}$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{k}$$

$\Rightarrow$ Total number of ways is

$$n\cdot(n-1)\cdot\ldots(n-k+1) = {}_nP_k$$

"n permute k"

$${}_nP_k = \frac{n!}{(n-k)!}$$

$$\frac{n\cdot(n-1)\ldots(n-k+1)\cdot\cancel{(n-k)}\ldots}{\cancel{(n-k)\cdot(n-k-1)}\ldots}$$

Example

$S = \{1, 2, 3\}$

How many ways are there to choose 2 elements without replacement & order matters?

$\underline{1}\ \underline{2}$

$\underline{1}\ \underline{3}$

$\underline{2}\ \underline{1}$ $\Longrightarrow$ 6 total ways

$\underline{2}\ \underline{3}$

$\underline{3}\ \underline{1}$

$\underline{3}\ \underline{2}$

$$\frac{n!}{(n-k)!} = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

(4) Unordered sampling without replacement

Consider a set S with n elements, we want to draw k such that order does not matter, and no replacement.

Second Rule of Counting

$$\#\text{ of ways to choose when order doesn't matter} = \frac{\#\text{ of ways to choose with order mattering}}{\#\text{ of ordered ways per unordered way.}}$$

So, $\frac{n!}{(n-k)!}$ gives k elements but order matters

There are $k!$ ways to rearrange the k elements.

choices: $\underline{k}$ $\underline{(k-1)}$ $\underline{(k-2)}$ $\underline{\ }$

So, unordered sampling w/out replacement

$\Rightarrow$ total number of ways is $\dfrac{\left\{\frac{n!}{(n-k)!}\right\}}{\{k!\}}$

Note: $\binom{n}{k} = \dfrac{n!}{(n-k)!\,(k)!}$

"n choose k"

Example

$S = \{1, 2, 3\}$

How many ways are there to choose 2 elements without replacement & order does not matter?

$3 \cdot 2$ $\quad \frac{3!}{(3-2)!}$

$\underline{1}\ \underline{2} \to \{1, 2\}$

$\underline{2}\ \underline{1} \to \{1, 2\}$

$\underline{1}\ \underline{3} \to \{1, 3\}$

$\underline{3}\ \underline{1} \to \{1, 3\}$

$\underline{2}\ \underline{3} \to \{2, 3\}$

$\underline{3}\ \underline{2} \to \{2, 3\}$

2! ordered ways per 1 unordered way

$$\binom{n}{k} = \binom{3}{2} = \frac{3!}{(3-2)!\,2!} = \frac{6}{1 \cdot 2!} = 3$$

$$= \frac{{}_nP_k}{k!}$$

(5) Unordered sampling with replacement.

Consider a set S with n elements, we want to draw k such that order does not matter and repitition is allowed.

Can try to use second rule of counting but run into issue of nonconstant number of ordered ways per unordered way.

Example:

$S = \{1, 2, 3\}$

1 → apples

2 → bananas

3 → oranges

unlimited supply

how many ways to select 5 pieces of fruit?

Can try second rule of counting:

↳ Pretend order matter

choices: $\underline{3} \cdot \underline{3} \cdot \underline{3} \; \underline{3} \; \underline{3} = 3^5$

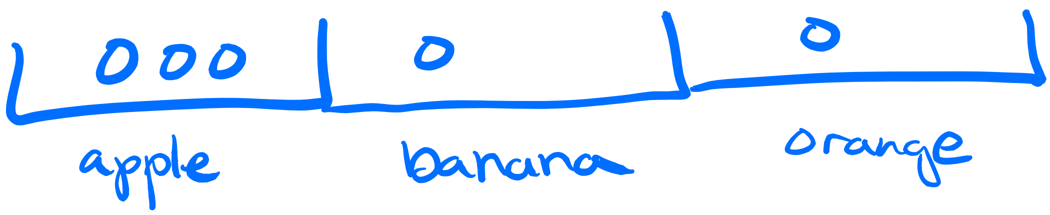
Then how many ordered ways for each unordered?

Ex: All bananas: 22222 → one way

1 apple, rest bananas. $\left.\begin{array}{l} 12222 \\ 21222 \\ 22122 \\ 22212 \\ 22221 \end{array}\right\}$ 5 ways

⇒ No easy way to map from ordered to unordered (no constant)

Bins perspective



↕ represent as a binary string

$\underline{0}\,\underline{0}\,\underline{0}\,\underline{1}\,\underline{0}\,\underline{1}\,\underline{0}$

"stars and bars"

How many such binary strings?

3 bins $\rightarrow$ 2 bars / dividers

5 fruits $\rightarrow$ 5 stars / balls

$$\frac{7!}{(7-2)!\,2!} = \binom{5+2}{2} = \binom{5+2}{5} = \frac{7!}{(7-5)!\,5!}$$

choose which slots to put the bars

choose which slots to put the stars

In general, $\binom{k+n-1}{k}$ ways

k = # choices / fruits

n = # possibilities for each choice

Zeroth rule of Counting

Want to count size of A?

find bijection between A & B

where B is easier to count, and just count size of B.

this works since $|A| = |B|$

6 Summary of Approaches

Number of ways to choose k items from set of size n

	replacement	no replacement
order matters	n^k	${}_nP_k = \frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k}$	${}_nC_k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$$\frac{(n+k-1)!}{(n+\cancel{k}-1-\cancel{k})!\;k!}$$

Break: 4:00 PM

Reminder:

Proctoring Gradescope Assignment HW2 Q1

7 Combinatorial Proofs

Idea: Use intuitive counting arguments instead of tedious algebraic manipulation to prove some identity.

Example

Prove $\binom{n}{k+1} = \binom{n-1}{k} + \binom{n-2}{k} + \dots + \binom{k}{k}$

Proof:

LHS : # of ways to choose $k+1$ element subset X from a set S containing n elements ($S = \{1, 2, \dots, n\}$)

RHS : Start by cases based on the lowest numbered element of X.

If lowest is 1, we choose k from the remaining $n-1$ elements.

If lowest is 2, we choose k from

the remaining $n-2$ elements of S

Proceeding in this way by cases depending on the ~~first~~ lowest numbered element X that we select, we sum the ways for each case.

$$\binom{n-1}{k} + \binom{n-2}{k} + \dots + \binom{k}{k}$$

This gives us all the ways to choose $k+1$ elements from set of n.

Since LHS and RHS count the same thing but in different ways, the two sides are equal.

Example

Prove $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

Proof:

Say you are constructing a binary string that is n bits long:

LHS:

You could make a string with no zeros
↳ $\binom{n}{0}$ ways

You could make a string with one zero.
↳ $\binom{n}{1}$ ways

⋮

Total: $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$ ways to make an n-bit binary string.

RHS:

Two choices for each position/slot/bit of the binary string, zero or one.

$$\underbrace{2\cdot 2\cdot 2\cdots 2}_{n \text{ times}} \Rightarrow 2^n \text{ binary strings.}$$

$n=3$

$k=1$

$\underline{0}\ \underline{1}\ \underline{1}$

$\underline{1}\ \underline{0}\ \underline{1}$

$\underline{1}\ \underline{1}\ \underline{0}$

$$\binom{n}{k} = \binom{3}{1} = 3$$