Rules of Counting

Zeroth Rule: For any sets A and B, if there exists $f: A \to B$ a bijection, then $|A| = |B|$ (definition)

First Rule: Suppose we are creating an object by k <u>successive</u> choices, where there are n^i options for the i^{th} choice. Then there are

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

possible objects.

Second Rule: Suppose we are creating an object by k choices, the order of which does not matter. If there exists an m-to-1 function between A, the set of objects created by ordered choices and B, the set of objects created by unorderd choices,

then

$$|B| = \frac{|A|}{m}$$

<u>Q</u> Let $S = \{1, 2, \ldots, n\}$ and $1 \leq k \leq n$.

How many sequences of k numbers from S are there?

How many sequences of k distinct numbers from S are there?

How many subsets of k distinct numbers from S are there?

How many "bags" of k numbers from S are there?

Combinatorial Proofs

Def A combinatorial proof counts some carefully chosen set in two different ways to show two expressions are equal.

Thm (Binomial)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Pf

Thm (Vandermonde's Identity)

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

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Permutations and Derangements

<u>Def</u> A permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is

<u>Ex</u>

<u>Q</u> How many permutations of $S = \{s_1, \ldots, s_n\}$ are there?

<u>Def</u> A derangement is

<u>Ex</u>

Derangements (Recursive)

Thm For $n \geq 3$, the number of derangements of $\{1, 2, \dots, n\}$, D_n, satisfies

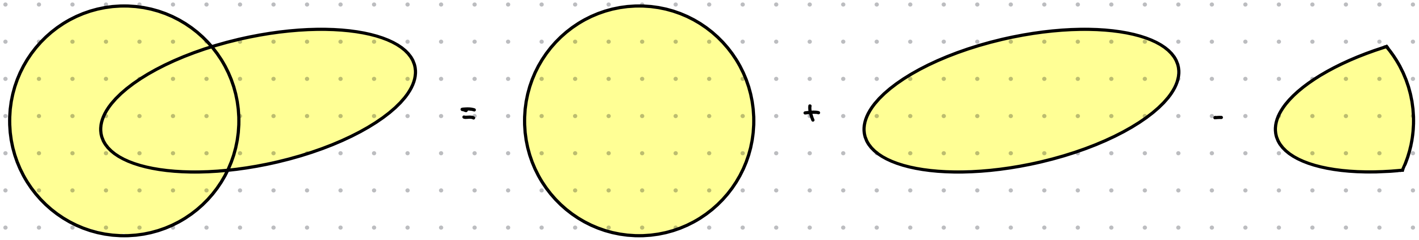
$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

Pf By combinatorial proof.

Deriving this recursion is not easy; nor is solving it. We will consider another approach.

Counting Unions

Q Consider any two sets A_1 and A_2. What is $|A_1 \cup A_2|$?



Q Consider any three sets A_1, A_2, A_3. What is $|A_1 \cup A_2 \cup A_3|$?

Ex Out of 50 animals

- 30 can fly $|A_1|$
- 12 can swim $|A_2|$
- 5 can fly and swim $|A_1 \cap A_2|$

How many animals can do at least one of flying and swimming?

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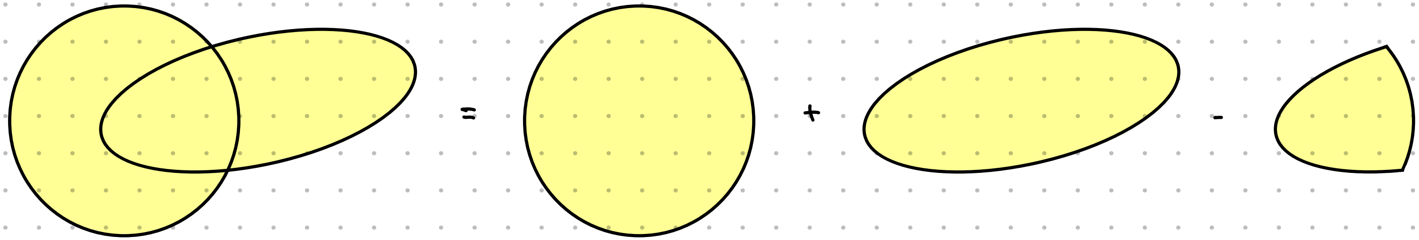

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The Principle of Inclusion - Exclusion

Thm (Inclusion-Exclusion) Let $A_1, \ldots, A_n$ be arbitrary finite subsets of some universal set A. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{i=1}^{n}\sum_{j=i}^{n}|A_i \cap A_j| + \sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{k=j}^{n}|A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} A_i\right|$$

Pf Combinatorially.

Note $$\left|\bigcup_{i=1}^{n} A_i\right| = \underbrace{\sum_{i=1}^{n}|A_i|}_{\text{singles}} - \underbrace{\sum_{i<j}|A_i \cap A_j|}_{\text{pairs}} + \underbrace{\sum_{i<j<k}|A_i \cap A_j \cap A_k|}_{\text{triples}} - \ldots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} A_i\right|$$

n terms $\quad \binom{n}{2}$ terms $\quad \binom{n}{3}$ terms $\quad \binom{n}{n}$ terms

The k^{th} summation has $\binom{n}{k}$ terms.

Note PIE allows us to express a union in terms of intersections.

Derangements (PIE)

Q What is a closed-form expression for D_n?