Rules of Counting

Zeroth Rule: For any sets A and B, if there exists $f: A \to B$ a bijection, then $|A| = |B|$ (definition)

First Rule: Suppose we are creating an object by k <u>successive</u> choices, where there are n^i options for the i^{th} choice. Then there are

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$

possible objects.

Second Rule: Suppose we are creating an object by k choices, the order of which does not matter. If there exists an m-to-1 function between A, the set of objects created by ordered choices and B, the set of objects created by unorderd choices, then

$$|B| = \frac{|A|}{m}$$

<u>Q</u> Let $S = \{1, 2, \ldots, n\}$ and $1 \le k \le n$.

How many sequences of k numbers from S are there?

$n \cdot n \cdot \ldots \cdot n = n^k$ (order matters, replacement)

How many sequences of k distinct numbers from S are there?

$$n \cdot (n-1) \cdot \ldots \cdot (n-(k-1)) = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-(k-1)) \cdot (n-k) \cdot \ldots \cdot 1}{(n-k) \cdot \ldots \cdot 1} = \frac{n!}{(n-k)!}$$ (order matters, no replacement)

How many subsets of k distinct numbers from S are there?

Each subset corresponds to $k!$ ordered sequences.

$$\frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

How many "bags" of k numbers from S are there?

E.g. for $n=3$, $k=3$, [1,1,1], [1,2,1], and [1,3,3] are distinct bags.

Consider how many of each element you get

★ ★ | ★ | … | ★★★ ($n-1$ bars, k stars)

1 2 … n

[1,1,1] ★ ★ ★ | |

[1,2,1] ★ ★ | ★ ★

[1,3,3] ★ | | ★ ★

There is a bijection from bags to stars + bars.

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$ We select the positions of the bars from $n+k-1$ total positions without replacement; the order in which we choose the bars does not matter.

Combinatorial Proofs

Def A combinatorial proof counts some carefully chosen set in two different ways to show two expressions are equal.

Thm (Binomial)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Pf Consider $(a+b)^n = (a+b)\cdot(a+b)\cdot\ldots\cdot(a+b)$. For each term $a^k b^{n-k}$, there are $\binom{n}{k}$ ways to choose the k factors $(a+b)$ that contribute to a^k (alternatively, $\binom{n}{n-k}$ ways to choose the $n-k$ factors $(a+b)$ that contribute to b^{n-k}).

Thm (Vandermonde's Identity)

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

Pf The RHS is the number of ways to choose r elements from $m+n$. Suppose our story is about choosing r people from m mages and n knights.

For the LHS, we can pick our r people by

(0) Picking 0 mages and r knights $\binom{m}{0}\binom{n}{r}$

(1) Picking 1 mage and $r-1$ knights $\binom{m}{1}\binom{n}{r-1}$

⋮ ⋮

(r) Picking r mages and 0 knights. $\binom{m}{r}\binom{n}{0}$

Permutations and Derangements

Def A permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ is a rearrangement of $S = \{s_1, s_2, \dots, s_n\}$ such that for any i, $\pi_i \in S$ and for any $i \neq j$, $\pi_i \neq \pi_j$.

Ex (B, A, C) is a permutation of $\{A, B, C\}$

Q How many permutations of $S = \{s_1, \dots, s_n\}$ are there?

To construct a permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$, we must select $\pi_1, \pi_2, \dots, \pi_n$ from S.

π_1 : n options

π_2 : $n-1$ options

$\vdots$

π_n : 1 option.

So there are $n \cdot (n-1) \cdot \ldots \cdot 1 = n!$ possible permutations.

Def A derangement is a permutation with no fixed points; that is, for any i, $\pi_i \neq s_i$

Ex (B, A, C) is not a derangement of $\{A, B, C\}$

The derangements of $\{1, 2, 3, 4\}$ are

π_1	π_2	π_3	π_4
2	1	4	3
4	1	2	3
2	4	1	3
3	4	1	2
4	3	1	2
3	1	4	2
4	3	2	1
3	4	2	1
2	3	4	1

Observations

① If $\pi_4 = j$ and $\pi_j = 4$, then the remaining $(n-2)$ elements must be deranged.

② If $\pi_4 = j$ and $\pi_j \neq 4$, then the remaining $(n-1)$ elements must be deranged.

Derangements (Recursive)

Thm For $n \geq 3$, the number of derangements of $\{1, 2, \dots, n\}$, D_n, satisfies
$$D_n = (n-1)(D_{n-1} + D_{n-2})$$
$\begin{pmatrix} D_1 = 0 \\ D_2 = 1 \end{pmatrix}$

Pf By combinatorial proof. For any derangement $\pi = (\pi_1, \pi_2, \dots, \pi_n)$, suppose $\pi_n = j \in \{1, 2, \dots, n-1\}$. There are $(n-1)$ such choices. For each choice of j, there are two cases.

① $\pi_j = n$. Then there is a bijection from the derangements of the remaining elements $\{1, 2, \dots, j-1, j+1, \dots, n-1\}$ and the derangements of $\{1, 2, \dots, n-2\}$. Therefore there are D_{n-2} derangements of the remaining elements. That is, we require
$$\pi_1 \neq 1, \quad \pi_2 \neq 2, \quad \dots, \quad \pi_{j-1} \neq j-1, \; \pi_{j+1} \neq j+1, \; \dots, \; \pi_{n-1} \neq n-1 \quad \text{for } \{1, 2, \dots, j-1, j+1, \dots, n-1\},$$
which, down to naming, is equivalent to
$$\tau_1 \neq 1, \quad \tau_2 \neq 2, \quad \dots, \quad \tau_{j-1} \neq j-1, \; \tau_j \neq j, \; \dots, \; \tau_{n-2} \neq n-2 \quad \text{for } \{1, 2, \dots, j, j+1, \dots, n-2\}.$$
By definition, there are D_{n-2} ways to create the derangement τ.

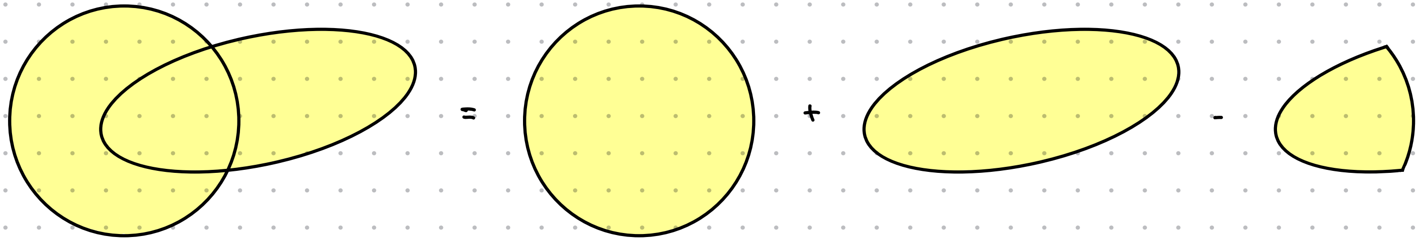
② $\pi_j \neq n$. Then there is a bijection from the derangements of the remaining elements $\{1, 2, \dots, j-1, j+1, \dots, n\}$ and the derangements of $\{1, 2, \dots, n-1\}$. Therefore there are D_{n-1} derangements of the remaining elements. That is, we require
$$\pi_1 \neq 1, \quad \pi_2 \neq 2, \quad \dots, \quad \pi_{j-1} = j-1, \; \pi_j \neq n, \; \pi_{j+1} \neq j+1, \; \dots, \; \pi_{n-1} \neq n-1 \quad \text{for } \{1, 2, \dots, j-1, j+1, \dots, n\},$$
which, down to naming, is equivalent to
$$\tau_1 \neq 1, \quad \tau_2 \neq 2, \quad \dots, \quad \tau_{j-1} \neq j-1, \; \tau_j \neq j, \; \dots, \; \tau_{n-1} \neq n-1 \quad \text{for } \{1, 2, \dots, j, j+1, \dots, n-1\}.$$
By definition, there are D_{n-1} ways to create the derangement τ.

So
$$D_n = \underset{\pi_n = j}{\underset{\uparrow}{(n-1)}}\,(\underset{\pi_j = n}{\underset{\uparrow}{D_{n-2}}} + \underset{\pi_j \neq n}{\underset{\uparrow}{D_{n-1}}})$$

Deriving this recursion is not easy; nor is solving it. We will consider another approach.

Counting Unions

Q Consider any two sets A_1 and A_2. What is $|A_1 \cup A_2|$?



$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. We count all the elements in A_1 and all the elements in A_2. Then each element of $A_1 \cap A_2$ is counted twice, so we subtract the count of all elements in $A_1 \cap A_2$.

Q Consider any three sets A_1, A_2, A_3. What is $|A_1 \cup A_2 \cup A_3|$?

We count all the elements in A_1, in A_2, and in A_3.

For $a \in A_i \cap A_j$ $(i \neq j)$, a was counted twice. Subtract $|A_1 \cap A_2|$, $|A_1 \cap A_3|$, $|A_2 \cap A_3|$.

For $a \in A_1 \cap A_2 \cap A_3$, a was counted thrice and subtracted thrice. Add $|A_1 \cap A_2 \cap A_3|$.

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| = &|A_1| + |A_2| + |A_3| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &+ |A_1 \cap A_2 \cap A_3|\end{aligned}$$

Ex Out of 50 animals

- 30 can fly $\quad |A_1|$
- 12 can swim $\quad |A_2|$
- 5 can fly and swim $\quad |A_1 \cap A_2|$

How many animals can do at least one of flying and swimming?

$30 + 12 - 5 = 37$

The Principle of Inclusion - Exclusion

Thm (Inclusion-Exclusion) Let $A_1, \dots, A_n$ be arbitrary finite subsets of some universal set A. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n}|A_i| - \sum_{i=1}^{n}\sum_{j=i+1}^{n}|A_i \cap A_j| + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}|A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1}\left|\bigcap_{i=1}^{n} A_i\right|$$

Pf Combinatorially. If $a \notin \bigcup A_i$, then $a \notin A_i$ for any A_i. So if a is not counted in the LHS, it is not counted in the RHS. Now consider $a \in \bigcup A_i$. We show that a is counted once in the RHS. Let

$M = \{i : a \in A_i\}$ $\quad m = |M|$

Then a is counted

$$m - \binom{m}{2} + \binom{m}{3} - \dots + \binom{m}{m} = \sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}$$

$$= (-1)\sum_{k=1}^{m}(-1)^{k}\binom{m}{k}$$

$$= (-1)\left(\sum_{k=0}^{m}(-1)^{k}\,1^{m-k}\binom{m}{k} - (-1)^{0}\binom{m}{0}\right)$$

$$= (-1)\left((-1+1)^{m} - 1\right)$$

$$= 1$$

So if a is counted in the LHS, then a is counted in the RHS.

Note $$\left|\bigcup_{i=1}^{n} A_i\right| = \underbrace{\sum_{i=1}^{n}|A_i|}_{\substack{\text{singles} \\ n \text{ terms}}} - \underbrace{\sum_{i<j}|A_i \cap A_j|}_{\substack{\text{pairs} \\ \binom{n}{2} \text{ terms}}} + \underbrace{\sum_{i<j<k}|A_i \cap A_j \cap A_k|}_{\substack{\text{triples} \\ \binom{n}{3} \text{ terms}}} - \dots + \underbrace{(-1)^{n-1}\left|\bigcap_{i=1}^{n} A_i\right|}_{\binom{n}{n} \text{ terms}}$$

The k^{th} summation has $\binom{n}{k}$ terms.

Note PIE allows us to express a union in terms of intersections.

Derangements (PIE)

Q What is a closed-form expression for D_n?

Counting the number of ways to have no fixed points is hard. It's much easier to count the number of ways to have fixed points. Let $F = \{\pi : \pi \text{ has at least one fixed point}\}$ and P be the set of permutations. Then

$$D_n = |P| - |F| = n! - |F|$$

Let $F_i = \{\pi : \pi_i = i\}$. Then $F = \bigcup F_i$, so

$$D_n = n! - \left|\bigcup_{i=1}^{n} F_i\right|$$

We can apply inclusion-exclusion

$|F_i| = (n-1)!$, since $\pi_i = i$

$|F_i \cap F_j| = (n-2)!$ since $\pi_i = i$, $\pi_j = j$

$\vdots$

$\left|\bigcap_{i=1}^{n} F_i\right| = 1$ since $\pi_1 = 1, \pi_2 = 2, \dots, \pi_n = n$

Therefore

$$\left|\bigcup_{i=1}^{n} F_i\right| = \sum_{i=1}^{n}(n-1)! - \sum_{i<j}(n-2)! + \dots + (-1)^{n-1} \cdot 1$$

$$= \binom{n}{1} \cdot (n-1)! - \binom{n}{2} \cdot (n-2)! + \dots + (-1)^{n-1}\binom{n}{n} \cdot (n-n)!$$

$$= \sum_{i=1}^{n} (-1)^{i-1}\binom{n}{i}(n-i)!$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!\cancel{(n-i)!}} \cancel{(n-i)!}$$

So

$$D_n = n! - \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!} = n! + \sum_{i=1}^{n} (-1)^{i} \frac{n!}{i!}$$

$$= n!\left(\frac{(-1)^0}{0!} + \sum_{i=1}^{n} \frac{(-1)^i}{i!}\right)$$

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$