Random Experiments

Def

Any sample point
An event is

Ex Consider the sample space of permutations of a, b, c.
$\Omega = \{abc, acb, bac, bca, cab, cba\}$
Any permutation, such as bca, is a sample point.
Consider the following events

A	"a is first"	$\{abc, acb, bca, cba\}$
B		
C	" is a derangement"	

Def A probability space
①
②

We define the probability of an event $A \subseteq \Omega$

Ex Suppose Ankit, Bob, and Chitra are randomly shuffled so that each permutation is equally likely.
What is the probability the order is abc?

What is the probability that Bob and Chitra are next to each other?

What is the probability they are deranged?

Equally Likely Outcomes I

Note If the sample space is uniform (all outcomes have the same probability), then

Q (Fair Coin) If I flip a fair coin 4 times, what is the probability all coin tosses are the same?

If I flip a fair coin 3 times, what is the probability I get exactly three heads?

If I flip a fair coin n times, what is the probability I get exactly $k \leq n$ heads?

Q (Fair Dice) Consider rolling two fair dice. What is a good outcome space for this experiment?

What is the probability that the sum is at least 10?

What is the probability that there is at least one 6?

Equally Likely Outcomes II

Q (Poker Hands): A deck consists of 52 cards of 13 ranks in each of 4 suits. What is the probability that a randomly shuffled poker hand is a flush (all 5 cards of the same suit)?

Q (Balls and Bins) I throw m balls into n bins such that each throw is equally likely to go into each of the bins and no throw affects the other throws.
What is the probability that all the balls go into the first bin?

What is the probability that none of the balls go into the first bin?

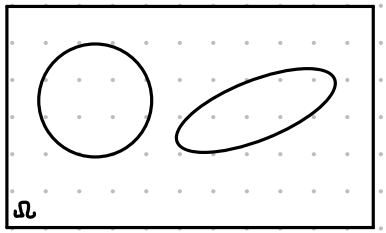
What is the probability that Bin 1 is not empty?

Note Balls and bins is an extremely powerful model! For example, flipping a fair coin 3 times is $m=3$, $n=2$. Rolling two dice is $m=2$, $n=6$.

Probability Rules

Rule (Addition) Suppose $A, B \subseteq \Omega$ and $A \cap B = \emptyset$. Then

$$P(A \cup B) = P(A) + P(B)$$



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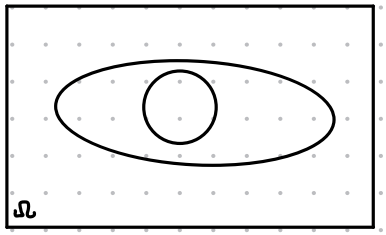
Rule (Difference) Suppose $A \subseteq \Omega$ and $B \subseteq A$. Then

$$P(A \setminus B) = P(A) - P(B)$$



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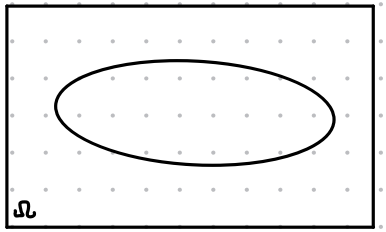
Rule (Complement) For any $B \subseteq \Omega$, $P(B) = 1 - P(\tilde{B})$



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Ex A fair die is rolled 3 times. What is the probability that at least one roll is greater than 1?

Examples

@ A fair coin is tossed n times. What is the chance of getting at least one head and at least one tail?

@ (Maximum) A fair die is rolled 10 times.
What is the chance the maximum is at most 5?

What is the chance that the maximum is at most 4?

What is the chance that the maximum is 5?

Birthday Paradox

Q Suppose there are n students in a room such that
- Each student's birthday is uniform over all days in the year;
- Each student's birthday does not affect any other student's birthday;
- No one is born on Leap Day.

What is the probability that at least two students have the same birthday?

Monty Hall Paradox

Q On a game show, there are 3 doors. Behind one door is a car, and behind the other two are goats. The following happens.

① The contestant picks a door

② One door is revealed to have a goat behind it.

③ The contestant has the option of staying or switching.

Should the contestant stay or switch (if they want the car)?