

## Random Experiments

Def The set of all possible outcomes of an experiment is called the sample space (or outcome space), and is denoted  $\Omega$ .

Any sample point (outcome) is denoted  $\omega \in \Omega$

An event is some set of sample points, denoted  $A \subseteq \Omega$ .

Ex Consider the sample space of permutations of  $a, b, c$ .

$$\Omega = \{abc, acb, bac, bca, cab, cba\}$$

Any permutation, such as  $bca$ , is a sample point.

Consider the following events

A	"a is first"	$\{abc, acb\}$
B	"b and c are next to each other"	$\{abc, acb, bac, cba\}$
C	"is a derangement"	$\{bca, cab\}$

Def A probability space is a sample space and a probability,  $P$  such that

①  $0 \leq P(\omega)$  for all  $\omega \in \Omega$  (nonnegative)

②  $\sum_{\omega \in \Omega} P(\omega) = 1$  (total one)

We define the probability of an event  $A \subseteq \Omega$  to be

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Ex Suppose Ankit, Bob, and Chitra are randomly shuffled so that each permutation is equally likely. What is the probability the order is  $abc$ ?

$P(abc) = \frac{1}{6}$ , since each of 6 outcomes is equally likely

What is the probability that Bob and Chitra are next to each other?

$$P(\{abc, acb, bac, cba\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

What is the probability they are deranged?

$$P(\{bca, cab\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

## Equally Likely Outcomes I

Note If the sample space is uniform (all outcomes have the same probability), then

$$P(\omega) = \frac{1}{|\Omega|} \quad P(A) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

Q (Fair Coin) If I flip a fair coin 4 times, what is the probability all coin tosses are the same? Each of the  $2^4$  sequences is equally likely.

$$P(\{HHHH, TTTT\}) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

If I flip a fair coin 3 times, what is the probability I get exactly three heads? The event is  $\{HHHT, HHTH, HTTH, THTH\}$ .

$$P(3 \text{ heads}) = \frac{4}{16} = \frac{1}{4}$$

If I flip a fair coin  $n$  times, what is the probability I get exactly  $k \leq n$  heads?

$$|\Omega| = 2 \cdot 2 \cdot \dots \cdot 2 = 2^n$$

$$|A| = \{\text{strings with } k \text{ Hs and } n-k \text{ Ts}\} = \binom{n}{k}$$

$$P(k \text{ heads}) = \frac{\binom{n}{k}}{2^n} = \binom{n}{k} \left(\frac{1}{2}\right)^n = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$

Q (Fair Dice) Consider rolling two fair dice. What is a good outcome space for this experiment?

$$\Omega = \{(i,j) : i, j \in \{1, \dots, 6\}\}$$

What is the probability that the sum is at least 10?

$$A = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$$

$$P(A) = \frac{6}{36}$$

What is the probability that there is at least one 6?

$$B = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

$$P(B) = \frac{11}{36}$$

## Equally Likely Outcomes II

Q (Poker Hands): A deck consists of 52 cards of 13 ranks in each of 4 suits. What is the probability that a randomly shuffled poker hand is a flush (all 5 cards of the same suit)?

Let  $\Omega$  be the set of possible 5-card hands. Then

$$|\Omega| = \binom{52}{5}$$

Each hand is equally likely, so we must count the number of hands that are all the same suit.

$$|A| = \binom{4}{1} \cdot \binom{13}{5}$$

↑                    ↑  
choose suit        choose 5 cards from suit

$$P(\text{Flush}) = \frac{4 \binom{13}{5}}{\binom{52}{5}}$$

Q (Balls and Bins) I throw  $m$  balls into  $n$  bins such that each throw is equally likely to go into each of the bins and no throw affects the other throws.

What is the probability that all the balls go into the first bin?

Let  $\Omega$  be the sequence of bins the balls land in. Then

$$P(\text{all in Bin 1}) = \frac{1}{n^m}$$

$P(\text{all in Bin 1}) \neq \frac{1}{\binom{n+m-1}{m}}$ , since the arrangements of balls into bins is not equally likely.

What is the probability that none of the balls go into the first bin?

$$P(\text{Bin 1 empty}) = \frac{(n-1)^m}{n^m} = \left(\frac{n-1}{n}\right)^m \quad P(\text{Bin 1 empty}) \neq 1 - \frac{1}{n^m}$$

What is the probability that Bin 1 is not empty?

$$P(\text{Bin 1 not empty}) = \frac{n^m - (n-1)^m}{n^m} \leftarrow \text{remove all sequences where Bin 1 is empty}$$
$$= 1 - \frac{(n-1)^m}{n^m}$$

Note Balls and bins is an extremely powerful model! For example, flipping a fair coin 3 times is  $m=3$ ,  $n=2$ . Rolling two dice is  $m=2$ ,  $n=6$ .

## Probability Rules

Rule (Addition) Suppose  $A, B \subseteq \Omega$  are events and  $A \cap B = \emptyset$ . Then

$$P(A \cup B) = P(A) + P(B)$$

pf Since  $A$  and  $B$  are disjoint,

$$P(A \cup B) = \sum_{\omega \in A \cup B} P(\omega) = \sum_{\omega \in A} P(\omega) + \sum_{\omega \in B} P(\omega) = P(A) + P(B)$$

Rule (Difference) Suppose  $A \subseteq \Omega$  and  $B \subseteq A$  are events. Then

$$P(A \setminus B) = P(A) - P(B)$$

pf Since  $B \subseteq A$ ,  $A = B \cup (A \setminus B)$  and  $B \cap (A \setminus B) = \emptyset$ . By addition,

$$P(B) + P(A \setminus B) = P(A), \text{ so } P(A \setminus B) = P(A) - P(B)$$

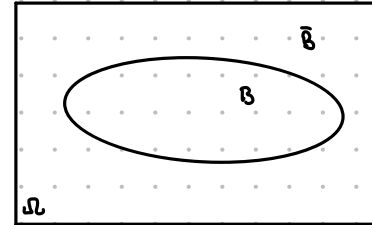
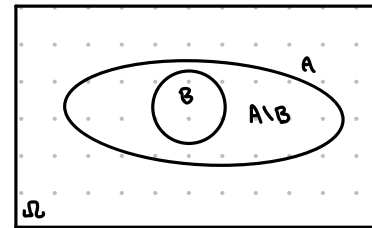
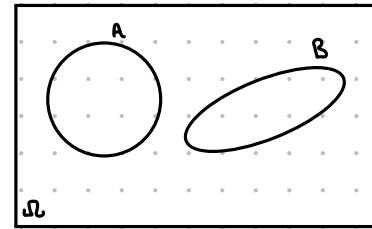
Rule (Complement) For any event  $B \subseteq \Omega$ ,  $P(B) = 1 - P(\bar{B})$

pf Let  $A = \Omega$ . By the difference rule,

$$P(\bar{B}) = P(\Omega \setminus B) = P(\Omega) - P(B) = 1 - P(B)$$

Ex A fair die is rolled 3 times. What is the probability that at least one roll is greater than 1?

$$\begin{aligned} P(\text{at least one roll} > 1) &= 1 - P(\text{all rolls} \leq 1) \\ &= 1 - P(\text{all rolls} = 1) \\ &= 1 - \frac{1}{6^3} \end{aligned}$$



## Examples

① A fair coin is tossed  $n$  times. What is the chance of getting at least one head and at least one tail?

We use the complement. There are many ways to get at least one of each, but many fewer ways to get no heads or no tails.

$$\begin{aligned}P(\text{at least 1 H, at least 1 T}) &= 1 - P(\text{no H or no T}) \\ &= 1 - P(n \text{ Tails or } n \text{ Heads}) \\ &= 1 - \frac{2}{2^n} = 1 - \frac{1}{2^{n-1}}\end{aligned}$$

② (Maximum) A fair die is rolled 10 times. What is the chance the maximum is at most 5?

If the maximum is at most 5, all 10 rolls can only be at most 5.

$$P(A) = \frac{5^{10}}{6^{10}}$$

What is the chance that the maximum is at most 4?

$$P(B) = \frac{4^{10}}{6^{10}}$$

What is the chance that the maximum is 5?

Note that if the maximum is at most 5, it is also at most 4:  $B \subseteq A$ .

Moreover, if the maximum is at most 5 and not at most 4, the maximum must be 5.

$$P(\text{max is 5}) = P(A \setminus B) = P(A) - P(B) = \frac{5^{10}}{6^{10}} - \frac{4^{10}}{6^{10}}$$

## Birthday Paradox

- Q Suppose there are  $n$  students in a room such that
- Each student's birthday is uniform over all days in the year;
  - Each student's birthday does not affect any other student's birthday;
  - No one is born on Leap Day.

What is the probability that at least two students have the same birthday?

Consider the complement: no two students have the same birthday. Then all the students must have distinct birthdays.

Let  $\omega = \{(b_1, \dots, b_n) : 1 \leq b_i \leq 365\}$ , i.e. our sample space is ordered sequences of birthdays.

$\bar{A}$  be the event that no students have the same birthday.

$$P(\bar{A}) = \frac{|\bar{A}|}{|\Omega|} = \frac{365 \cdot (365-1) \cdot \dots \cdot (365-(n-1))}{365^n} = \frac{365!}{365^n (365-n)!}$$

So

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{365!}{365^n (365-n)!}$$

Why did we use an ordered sample space? Because we need all outcomes to be equally likely. Since we are sampling with replacement, outcomes are not equally likely:

$$P(\text{Jan 1, Jan 1}) = \frac{1}{365^2}$$

$$P(\text{Jan 1, Jan 2}) = \frac{2}{365^2} \leftarrow \text{two ways for two people to have Jan 1 and Jan 2}$$

So we cannot use

$$P(A) = \frac{|A|}{|\Omega|}$$

if we want to use an unordered sample space. Instead we must use

$$P(A) = \sum_{\omega \in A} P(\omega)$$

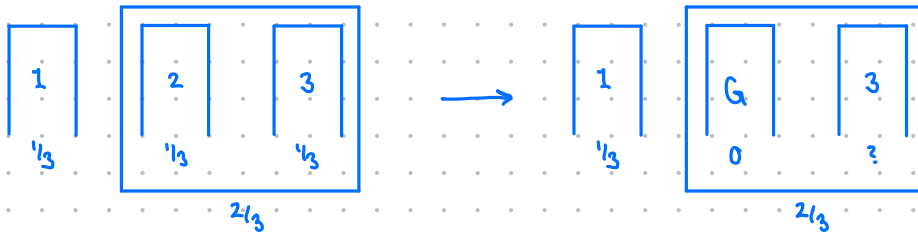
# Monty Hall Paradox

① On a game show, there are 3 doors. Behind one door is a car, and behind the other two are goats. The following happens.

- ① The contestant picks a door
- ② One door is revealed to have a goat behind it.
- ③ The contestant has the option of staying or switching.

Should the contestant stay or switch (if they want the car)?

Intuitively, it seems like it doesn't matter. However, it is actually better to switch.



The chance of winning the car by switching is  $\frac{2}{3}$ .

In the extreme case, consider 100 doors, 99 goats and a car.

- ① Contestant selects a door
- ② 98 doors are revealed to be goats
- ③ The contestant has the option of staying or switching.

Let the outcome space  $\Omega$  be the tuples (Prize door, Contestant door, Revealed door). Then if the contestant chooses door 1, the tuples are

- (1, 1, 2)      (2, 1, 3)      (3, 1, 2)  
(1, 1, 3)

So there are 4 tuples for each contestant choice and 12 total. Then

$$\begin{aligned} \textcircled{1} \quad P((2, 1, 3)) &= \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9} \\ \textcircled{2} \quad P((1, 1, 2)) &= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18} \end{aligned} \quad \left. \begin{array}{l} (2, 1, 3) \quad (1, 2, 3) \\ (3, 1, 2) \quad (3, 2, 1) \\ (1, 1, 2) \quad (2, 2, 1) \\ (1, 1, 3) \quad (2, 2, 3) \end{array} \right\} \begin{array}{l} (2, 3, 1) \\ (1, 3, 2) \\ (3, 3, 1) \\ (3, 3, 2) \end{array}$$

Note: we are multiplying probabilities with no justification.

There is a  $\frac{6}{9} = \frac{2}{3}$  probability of case 1 (switching is good) and a  $\frac{6}{18} = \frac{1}{3}$  probability of case 2 (switching is bad)