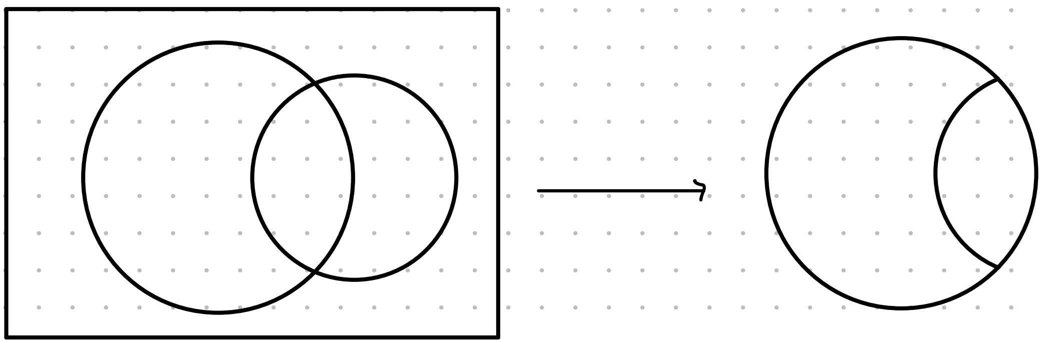
Conditional Probability

Rule (Division) Let $A, B \subseteq \Omega$ be events such that $\mathbb{P}(A) \neq 0$. Then the conditional probability of B given A, denoted $\mathbb{P}(B \mid A)$, is

Pf



Rule (Multiplication) For any events $A, B \subseteq \Omega$,

Generally, for events $A_1, \ldots, A_n \subseteq \Omega$,

Examples

Q Suppose $m = 4$ balls are thrown into $n = 3$ labeled bins. What is the probability that the first bin is empty given the second bin is empty?

Q What is the probability that the second card drawn from a standard deck is an ace given that the first card drawn is an ace?

Bayes Rule

<u>Rule</u> (Bayes) Let $A, B \subseteq \Omega$ be events such that $\mathbb{P}(B) \neq 0$. Then

<u>Pf</u> By multiplication and division rules.

<u>Ex</u> The incidence of a disease in the population is 5%. A test for the disease is such that

- If infected, the chance of a positive is 90%;
- If uninfected, the chance of a negative is 80%.

If a test returns positive, what is the chance the tested person has the disease?

(Drawing not to scale)

Law of Total Probability

Note Previously, we were able to partition an event by looking at the complement of another event:

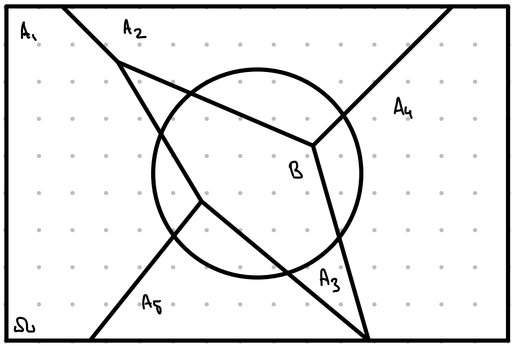
$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap \bar{A})$$

Rule (Law of Total Probability) Suppose $A_1, \dots, A_n$ partition the outcome space Ω. That is,

1.

2.

Then

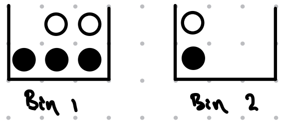


By adding up the intersections of B with the partitions, we can recover B

Ex A box contains 3 gold coins and 7 silver coins. Two are drawn at random without replacement. What is the chance that one of each kind is drawn?

Examples

Q Consider the following bins



A bin is picked uniformly at random and a ball is drawn from it. What is the chance we picked Bin 1 given the ball is white?

Q (Polya's Urn) An urn contains b black balls and w white balls. A ball is drawn independently and uniformly at random and replaced with r balls of its color. The process continues.
What is the chance the first draw is black?

What is the chance the second ball drawn is black given the first ball drawn is black?

Examples

Q (Polya's Urn) An urn contains b black balls and w white balls. A ball is drawn independently and uniformly at random and replaced with r balls of its color. The process continues.
What is the chance the second draw is black?

What is the chance the first ball drawn was black given the second is black?

Independence

Def Two events $A, B \subseteq \Omega$ are said to be independent, denoted $A \perp\!\!\!\perp B$, if

That is, information about one doesn't affect the information of the other.

Claim The above definition is equivalent to

Pf Exercise

Def Events $A_1, \dots, A_n \subseteq \Omega$ are said to be (mutually) independent if every possible subset of $\{A_1, \dots, A_n\}$ is independent:

Ex Suppose 2 balls are thrown into 2 bins with uniform and independent probabilities.
Consider the two outcome spaces Ω_1 and Ω_2.

$\Omega_1 = \{(2,0), (1,1), (0,2)\}$
(resulting arrangement of balls)

$\Omega_2 = \{11, 12, 21, 22\}$
(sequence of throws)

Note In lecture yesterday, I tried to do a version of the birthday paradox without order. This doesn't work!

Examples (& Symmetry)

Q Let A_1 and A_2 be the events that the first and second cards drawn from a standard deck are aces. Are the events independent?

Q A six-sided die is rolled n times. Let $A_1, \dots, A_6$ be the events that faces $1, \dots, 6$, respectively appear. Are the events $A_1, \dots, A_6$ independent?

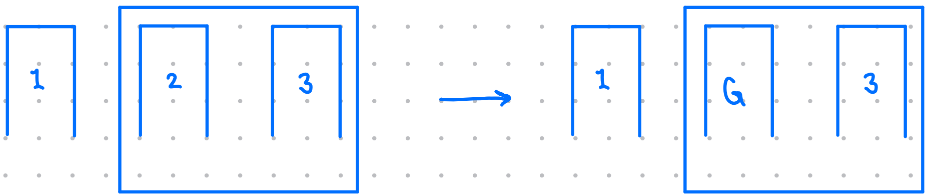
Q Consider two flips of a fair coin. Let H_1, H_2 be the events that the first and second tosses are heads, and S be the event that both tosses are the same.

Monty Hall Paradox

Q On a game show, there are 3 doors. Behind one door is a car, and behind the other two are goats. The following happens.

1. The contestant picks a door
2. One door is revealed to have a goat behind it.
3. The contestant has the option of staying or switching.

Should the contestant stay or switch (if they want the car)?



Unions of Events

Thm (Principle of Inclusion-Exclusion) For any events $A_1, \dots, A_n \subseteq \Omega$,

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \underbrace{\sum_{i=1}^n \mathbb{P}(A_i)}_{\text{singles}} - \underbrace{\sum_{i=1}^n \sum_{j=i+1}^n \mathbb{P}(A_i \cap A_j)}_{\text{pairs}} + \underbrace{\sum_{i=1}^n \sum_{j=i+1}^n \sum_{k=i+1}^n \mathbb{P}(A_i \cap A_j \cap A_k)}_{\text{triples}} - \dots + (-1)^{n-1} \mathbb{P}\left(\bigcap_{i=1}^n A_i\right)$$

Pf By induction.

Ex Suppose n letters are placed into n envelopes independently and uniformly at random. What is the chance that no letter goes into the correct location?

Thm (Boole's Inequality, Union Bound) For any events $A_1, \dots, A_n \subseteq \Omega$,

Pf By induction.

Note Complementation allows us to switch between unions and intersections:

Always consider all possible approaches.

Q A bag has 10 red marbles, 5 blue marbles, and 15 green marbles. Uniformly at random, n marbles are sampled with replacement from the bag. What is the probability that all colors appear?