Conditional Probability

Rule (Division) Let $A, B \subseteq \Omega$ be events such that $\mathbb{P}(A) > 0$. Then the conditional probability of B given A, denoted $\mathbb{P}(B \mid A)$, is

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$

Pf The outcomes $\omega \in A$ are the only possible outcomes, given A occurs. However,

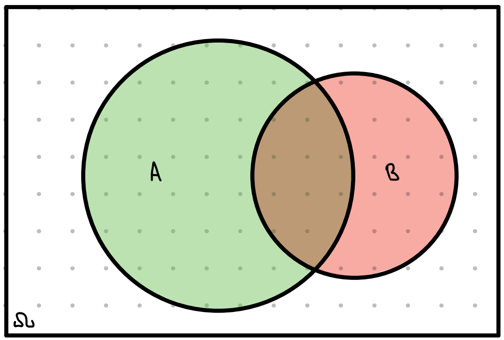
$$\sum_{\omega \in A} \mathbb{P}(\omega) = \mathbb{P}(A) \neq 1$$

So we must renormalize:

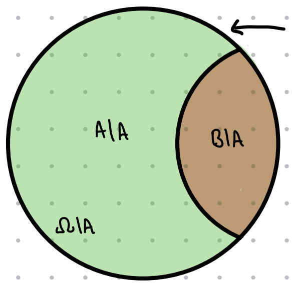
$$\mathbb{P}(\omega \mid A) = \frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}$$

Then

$$\mathbb{P}(B \mid A) = \sum_{\omega \in A \cap B} \frac{\mathbb{P}(\omega)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)}$$



Given A, we remove any of Ω that can no longer be the case



A becomes the new outcome space

Rule (Multiplication) For any events $A, B \subseteq \Omega$,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) = \mathbb{P}(B) \cdot \mathbb{P}(B \mid A)$$

Generally, for events $A_1, \dots, A_n \subseteq \Omega$,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 \mid A_1) \cdot \mathbb{P}(A_3 \mid A_1 \cap A_2) \cdot \ldots \cdot \mathbb{P}(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Examples

Q Suppose $m=4$ balls are thrown into $n=3$ labeled bins. What is the probability that the first bin is empty given the second bin is empty?

Let E_1 and E_2 be the events that bins 1 and bins 2 are empty. Then

$$\mathbb{P}(E_1 \mid E_2) = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_2)}$$

$$\mathbb{P}(E_1 \cap E_2) = \frac{(n-2)^m}{n^m} = \left(\frac{n-2}{n}\right)^m$$

$$\mathbb{P}(E_2) = \frac{(n-1)^m}{n^m} = \left(\frac{n-1}{n}\right)^m$$

So

$$\mathbb{P}(E_1 \mid E_2) = \left(\frac{n-2}{n-1}\right)^m$$

Alternatively, note that if Bin 2 must be empty, then we are throwing m balls into $n-1$ bins.

Note: we had to use an ordered outcome space since while any sequence of throws is equally likely, any ending arrangement of balls is not.

Q What is the probability that the second card drawn from a standard deck is an ace given that the first card drawn is an ace?

Let A_1 and A_2 be the events that the first and second cards are aces.

Then

$$\mathbb{P}(A_2 \mid A_1) = \frac{3}{51} \begin{array}{l} \leftarrow \text{number of aces} \\ \leftarrow \text{possible outcomes} \end{array}$$

Alternatively,

$$\mathbb{P}(A_2 \mid A_1) = \frac{\mathbb{P}(A_2 \cap A_1)}{\mathbb{P}(A_1)} = \frac{\frac{4 \cdot 3}{52 \cdot 51}}{\frac{4}{52}} = \frac{3}{51} = \frac{\binom{4}{2}/\binom{52}{2}}{\binom{4}{1}/\binom{52}{1}}$$

Note: we could use either an ordered or an unordered outcome space, since all outcomes were equally likely in both cases.

Bayes Rule

Rule (Bayes) Let $A, B \subseteq \Omega$ be events such that $\mathbb{P}(B) \neq 0$. Then

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$$

Pf By multiplication and division rules.

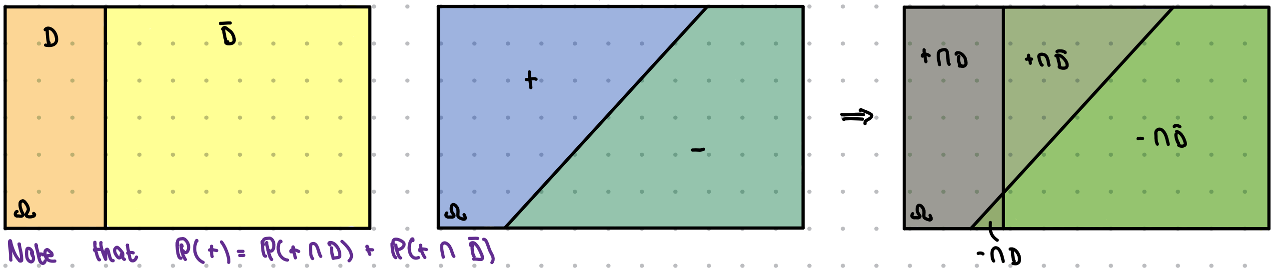
Ex The incidence of a disease in the population is 5%. A test for the disease is such that
- If infected, the chance of a positive is 90%;
- If uninfected, the chance of a negative is 80%.

If a test returns positive, what is the chance the tested person has the disease?

Know: $\mathbb{P}(+ \mid D) = 90\%$ $\quad \mathbb{P}(- \mid D) = 1 - 90\% = 10\%$ $\quad \mathbb{P}(D) = 5\%$

$\mathbb{P}(- \mid \bar{D}) = 80\%$ $\quad \mathbb{P}(+ \mid \bar{D}) = 1 - 80\% = 20\%$ $\quad \mathbb{P}(\bar{D}) = 95\%$

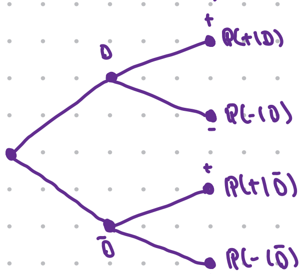
Want: $$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(+ \mid D) \cdot \mathbb{P}(D)}{\mathbb{P}(+)} = \frac{(.9)(.05)}{\mathbb{P}(+)}$$



(Drawing not to scale)

Note that
$$\begin{aligned}\mathbb{P}(+) &= \mathbb{P}(+ \cap D) + \mathbb{P}(+ \cap \bar{D}) \\ &= \mathbb{P}(+ \mid D) \cdot \mathbb{P}(D) + \mathbb{P}(+ \mid \bar{D}) \cdot \mathbb{P}(\bar{D}) \\ &= (.9)(.05) + (.2)(.95)\end{aligned}$$

$$\mathbb{P}(D \mid +) = \frac{(.9)(.05)}{(.9)(.05) + (.2)(.95)} \approx 0.19$$



Law of Total Probability

Note Previously, we were able to partition an event by looking at the complement of another event:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap \bar{A})$$

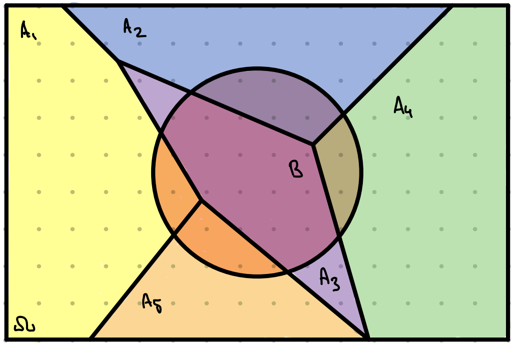
Rule (Law of Total Probability) Suppose $A_1, \ldots, A_n$ partition the outcome space Ω. That is,

① $\bigcup_{i=1}^{n} A_i = \Omega$

② $A_i \cap A_j = \emptyset$ for $i \neq j$ (that is, each pair is mutually exclusive).

Then

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i) \cdot \mathbb{P}(A_i)$$



By adding up the intersections of B with the partitions, we can recover B

$$\mathbb{P}(B) = \mathbb{P}(B \cap A_1) + \mathbb{P}(B \cap A_2) + \mathbb{P}(B \cap A_3) + \mathbb{P}(B \cap A_4) + \mathbb{P}(B \cap A_5)$$

$$= \sum_{i=1}^{5} \mathbb{P}(B \cap A_i)$$

$$= \sum_{i=1}^{5} \mathbb{P}(B \mid A_i) \cdot \mathbb{P}(A_i)$$

Ex A box contains 3 gold coins and 7 silver coins. Two are drawn at random without replacement. What is the chance that one of each kind is drawn?

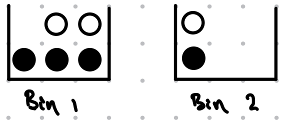
$$\mathbb{P}(\text{one of each kind}) = \mathbb{P}(\text{one of each kind} \mid \text{gold}) \cdot \mathbb{P}(\text{gold}) + \mathbb{P}(\text{one of each kind} \mid \text{silver}) \cdot \mathbb{P}(\text{silver})$$

$$= \frac{7}{9} \cdot \frac{3}{10} + \frac{3}{9} \cdot \frac{7}{10}$$

$$= 2\left(\frac{7 \cdot 3}{9 \cdot 10}\right) = \frac{\binom{3}{1}\binom{7}{1}}{\binom{10}{2}}$$

Examples

Q Consider the following bins



A bin is picked uniformly at random and a ball is drawn from it. What is the chance we picked Bin 1 given the ball is white?

$$\mathbb{P}(B_1 \mid W) = \frac{\mathbb{P}(B_1 \cap W)}{\mathbb{P}(W)}$$

$$\mathbb{P}(B_1 \cap W) = \mathbb{P}(W \mid B_1) \cdot \mathbb{P}(B_1)$$
$$= \quad 2/5 \cdot 1/2 = 2/10$$

$$\mathbb{P}(W) = \mathbb{P}(W \cap B_1) + \mathbb{P}(W \cap B_2)$$
$$= \mathbb{P}(W \mid B_1) \cdot \mathbb{P}(B_1) + \mathbb{P}(W \mid B_2) \cdot \mathbb{P}(B_2)$$
$$= \quad 2/10 \quad + \quad 1/2 \cdot 1/2 = 2/10 + 1/4$$

So

$$\mathbb{P}(B_1 \mid W) = \frac{2/10}{2/10 + 1/4} = \frac{4}{9}$$

Q (Polya's Urn) An urn contains b black balls and w white balls. A ball is drawn independently and uniformly at random and replaced with r balls of its color. The process continues.
What is the chance the first draw is black?

$$\mathbb{P}(B_1) = \frac{b}{b+w}$$

What is the chance the second ball drawn is black given the first ball drawn is black?

$$\mathbb{P}(B_2 \mid B_1) = \frac{b+d}{b+w+d}$$

Examples

Q (Polya's Urn) An urn contains b black balls and w white balls. A ball is drawn independently and uniformly at random and replaced with r balls of its color. The process continues.
What is the chance the second draw is black?

$$\mathbb{P}(B_2) = \mathbb{P}(B_1)\cdot \mathbb{P}(B_2 \mid B_1) + \mathbb{P}(\bar{B}_1)\,\mathbb{P}(B_2 \mid \bar{B}_1)$$

$$= \frac{b}{b+w}\cdot\frac{b+d}{b+w+d} + \frac{w}{b+w}\cdot\frac{b}{b+w+d} = \frac{b^2+db+wb}{(b+w)(b+w+d)} = \frac{b(b+w+d)}{(b+w)(b+w+d)}$$

$$= \frac{b}{b+w}$$

What is the chance the first ball drawn was black given the second is black?

$$\mathbb{P}(B_1 \mid B_2) = \frac{\mathbb{P}(B_1 \cap B_2)}{\mathbb{P}(B_2)}$$

$$= \frac{\frac{b}{b+w}\cdot\frac{b+d}{b+w+d}}{\frac{b}{b+w}}$$

$$= \frac{b+d}{b+w+d}$$

The model is symmetric in time!

Independence

Def Two events $A, B \subseteq \Omega$, $\mathbb{P}(A) > 0$, are said to be independent, denoted $A \perp\!\!\!\perp B$, if
$$\mathbb{P}(B \mid A) = \mathbb{P}(B)$$
That is, information about one doesn't affect the information of the other.

Claim The above definition is equivalent to
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Pf Exercise

Ex Suppose 2 balls are thrown into 2 bins with uniform and independent probabilities.
Consider the two outcome spaces Ω_1 and Ω_2.

$\Omega_1 = \{(2,0), (1,1), (0,2)\}$
(resulting arrangement of balls)

$$\left.\begin{aligned} \mathbb{P}(2,0) &= \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \\ \mathbb{P}(1,1) &= \tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{2}{4} \\ \mathbb{P}(0,2) &= \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \end{aligned}\right\}\text{ not equally likely}$$

$\Omega_2 = \{11, 12, 21, 22\}$
(sequence of throws)

$$\left.\begin{aligned} \mathbb{P}(11) &= \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \\ \mathbb{P}(12) &= \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \\ \mathbb{P}(21) &= \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \\ \mathbb{P}(22) &= \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4} \end{aligned}\right\}\text{ equally likely outcomes}$$

Note In lecture yesterday, I tried to do a version of the birthday paradox without order. This doesn't work!
$|\Omega| \neq \frac{365^n}{n!}$, as this isn't even an integer

Since the birthday paradox is with replacement, we can't do unordered counting and have equally likely outcomes! We would have to use the method of adding the probabilities of each outcome.
As a rule of thumb, ordered/unordered doesn't matter when sampling without replacement (since ${}_nP_k = \binom{n}{k}/k!$), but it does matter when sampling with replacement.

Def Events $A_1, \dots, A_n \subseteq \Omega$ are said to be (mutually) independent if every possible subset of $\{A_1, \dots, A_n\}$ is independent:
$$\mathbb{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathbb{P}(A_i) \quad \text{for any } S \subseteq \{1, \dots, n\}$$
Note that if all the A_i are pairwise independent, it is not necessarily the case that all the A_i are mutually independent.

Examples

Q Let A_1 and A_2 be the events that the first and second cards drawn from a standard deck are aces. Are the events independent?

Intuitively no; getting an ace on the first draw affects the chance of an ace on the second draw.
We must check whether

$$\mathbb{P}(A_2 \mid A_1) = \mathbb{P}(A_2)$$

From earlier,

$$\mathbb{P}(A_2 \mid A_1) = 3/51$$

Now we calculate $\mathbb{P}(A_2)$

$$\mathbb{P}(A_2) = \mathbb{P}(A_2 \mid A_1)\cdot \mathbb{P}(A_1) + \mathbb{P}(A_2 \mid \bar{A}_1)\cdot \mathbb{P}(\bar{A}_1)$$

$$= \frac{3}{51}\cdot\frac{4}{52} + \frac{4}{51}\cdot\frac{48}{52}$$

$$= \frac{4}{52}\left(\frac{3}{51} + \frac{48}{51}\right) = \frac{4}{52}$$

So $\mathbb{P}(A_2 \mid A_1) \neq \mathbb{P}(A_2)$

However, note that $\mathbb{P}(A_1) = \mathbb{P}(A_2)$. In fact, by induction,

$$\mathbb{P}(A_1) = \mathbb{P}(A_2) = \dots = \mathbb{P}(52) = 4/52$$

This is known as symmetry in simple random sampling; even though the draws are dependent, when without information about the other draws, the probabilities for each draw are the same.

Q A six-sided die is rolled n times. Let $A_1, \dots, A_6$ be the events that faces $1, \dots, 6$, respectively appear. Are the events $A_1, \dots, A_6$ independent?

No. If none of faces 1 through 5 appear, it must be the case that face 6 appears.

$$\mathbb{P}(A_6 \mid \bar{A}_1 \cap \bar{A}_2 \cap \dots \bar{A}_5) = 1 \neq \mathbb{P}(A_6)$$

Q Consider two flips of a fair coin. Let H_1, H_2 be the events that the first and second tosses are heads, and S be the event that both tosses are the same.

$H_1 \perp\!\!\!\perp H_2$

$H_1 \perp\!\!\!\perp S$: $\mathbb{P}(S \mid H_1) = \frac{1}{2} = \mathbb{P}(S)$

$H_2 \perp\!\!\!\perp S$: $\mathbb{P}(S \mid H_2) = \frac{1}{2} = \mathbb{P}(S)$

H_1, H_2, and S not mutually independent: $\mathbb{P}(S \mid H_1 \cap H_2) = 1 \neq \mathbb{P}(S)$

Monty Hall Paradox

Q On a game show, there are 3 doors. Behind one door is a car, and behind the other two are goats. The following happens.

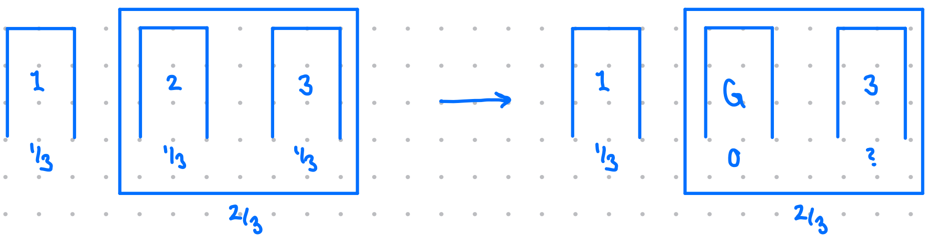
① The contestant picks a door

② One door is revealed to have a goat behind it.

③ The contestant has the option of staying or switching.

Should the contestant stay or switch (if they want the car)?

Intuitively, it seems like it doesn't matter. However, it is actually better to switch.



Let P_i be the event that the prize is behind door i,

C_i be the event that the contestant chooses door i,

H_i be the event that the host reveals door i.

Then

$$P(P_2 \mid C_1 \cap H_3) = \frac{P(P_2 \cap C_1 \cap H_3)}{P(C_1 \cap H_3)}$$

$$P(P_2 \cap C_1 \cap H_3) = \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$$

host must reveal door 3

$$P(C_1 \cap H_3) = P(C_1) \cdot P(H_3 \mid C_1) = \frac{1}{3} \cdot \frac{1}{2}$$

host can reveal either door 2 or 3

$$P(P_2 \mid C_1 \cap H_3) = \frac{1/9}{1/6} = \frac{2}{3}$$

So there is a $2/3$ chance that switching is superior.

Unions of Events

Thm (Principle of Inclusion-Exclusion) For any events $A_1, \dots, A_n \subseteq \Omega$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \underbrace{\sum_{i=1}^{n} \mathbb{P}(A_i)}_{\text{singles}} - \underbrace{\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathbb{P}(A_i \cap A_j)}_{\text{pairs}} + \underbrace{\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i+1}^{n} \mathbb{P}(A_i \cap A_j \cap A_k)}_{\text{triples}} - \dots + (-1)^{n-1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$$

Pf By induction.

Ex Suppose n letters are placed into n envelopes independently and uniformly at random. What is the chance that no letter goes into the correct location?

From Lecture 4A, there are $n!$ permutations and D_n derangements.

$$\mathbb{P}(\text{derangement}) = \frac{n! \sum_{k=1}^{n} \frac{(-1)^k}{k!}}{n!} = \sum_{i=1}^{n} \frac{(-1)^k}{k!} \to \frac{1}{e} \quad \text{as } n \to \infty$$

Thm (Boole's Inequality, Union Bound) For any events $A_1, \dots, A_n \subseteq \Omega$,

$$\max_i(\mathbb{P}(A_i)) \le \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Pf By induction.

Note Complementation allows us to switch between unions and intersections:

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i} \qquad \overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$$

Always consider all possible approaches.

Q A bag has 10 red marbles, 5 blue marbles, and 15 green marbles. Uniformly at random, n marbles are sampled with replacement from the bag. What is the probability that all colors appear?

Let the events R, B, and G be the events that we see a red, blue, and green marble, respectively.

Consider

- Are the events independent?
- Is it easy to find $\mathbb{P}(B \mid R)$ and $\mathbb{P}(G \mid R \cap B)$?

Then, consider the event $\bar{R} \cup \bar{B} \cup \bar{G}$.