0 Announcements

Midterm scores are out!

Regrade requests:
Estimated grade bins

Midsemester survey
Please let us know how things are going.

One-on-one meetings
If you want to talk to a TA about anything related to this class, fill out the form

A comment.
Exams and grades

A reminder:

Monday HW session →

HW problems take time!

You can sign up for a small discussion section at any time during the summer.

(0.5) Recap

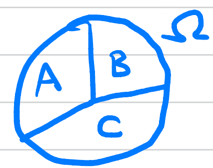
Previously on CS70...

Conditional probability $P(A|B) =$

Bayes' rule $P(A|B) =$

Simple proof. Profound implications?

Law of total probability



$P(A) + P(B) + P(C) =$

① Independence

<u>Def</u> Events A and B are independent if

Intuitive meaning:

Formal version of intuition

<u>Prop</u> If $P(B) > 0$ then A & B are independent if and only if

<u>pf</u> $(\Rightarrow)$ $P(A \cap B) = P(A)P(B) \Rightarrow$

$(\Leftarrow)$ $P(A | B) = P(A) \Rightarrow$

<u>Cor</u> If $P(A), P(B) > 0$ then

Def Events A and B are independent if $P(A \cap B) = P(A)P(B)$

Intuitive meaning: Knowing whether or not A occurred tells you nothing about whether B occurred

Example Flip a fair coin and then flip it again

A = 1^{st} flip is H

B = 1^{st} flip is H

Independence is often an assumption you make when

Example Flip a fair coin 2 times

A = 1^{st} flip is H

B = both flips are H

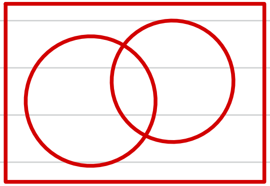
Question Suppose A and B are independent. What about A and $\overline{B}$?

Answer

Intuitively

Prop If A and B are independent events then

pf



② Pairwise vs. Mutual Independence

Question Suppose A, B, C are events

A, B independent

A, C independent

B, C independent

What is $P(A \cap B \cap C)$?

Answer

Example Flip a fair coin 2 times

A = 1st flip is H

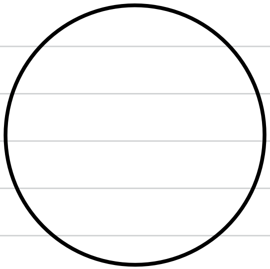
B = 2nd flip is H

C =

$P(A) =$

$P(B) =$

$P(C) =$



$A \cap B =$ $\quad P(A \cap B) =$

$A \cap C =$ $\quad P(A \cap C) =$

$B \cap C =$ $\quad P(B \cap C) =$

$P(A \cap B \cap C) =$

Def Events $A_1, A_2, \ldots, A_n$ are pairwise independent

if

Def Events $A_1, A_2, \ldots, A_n$ are mutually independent

if

What does this mean for $n = 3$?

Note: Mutual independence   Pairwise independence
Pairwise independence   Mutual independence

A familiar example...

Suppose you want to share a secret $s \in \{0, 1, 2, \dots, 10\}$

Generate a polynomial $p(x) = a_3 x^3 + a_2 x^2 + a_1 x + s$ over $GF(11)$
by picking a_3, a_2, a_1 randomly from $\{0, 1, 2, \dots, 10\}$

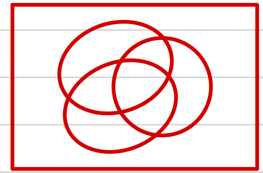
Hand out shares $p(x_0), p(x_1), \dots, p(x_n)$ $\quad x_i \neq x_j \quad x_i \neq 0$

Define $A_i =$

<u>Question</u>

<u>Answer</u>

Question Suppose A, B, C mutually independent.
What is $P(A \cap B \cap \overline{C})$?



Answer

Why?

Prop $A_1, A_2, \ldots, A_n$ are mutually independent if and only if

③ <u>Mutually Exclusive Events</u>

<u>Def</u> Events A and B are mutually exclusive if

A, B mutually exclusive $\Rightarrow$ $P(A \cap B) =$
$P(A \cup B) =$

<u>Example</u> Flip a fair coin 2 times
A =
B =

<u>Question</u> Suppose A, B are independent events with $P(A), P(B) > 0$. Can A & B be mutually exclusive?

<u>Answer</u>

Warning: Mutually exclusive events are (almost) never independent!!

④ Combinations of Events

Given events $A_1, A_2, \ldots, A_n$, how do we calculate $P(A_1 \cap A_2 \cap \ldots A_n)$ and $P(A_1 \cup A_2 \cup \ldots A_n)$?

Examples

① You are in charge of maintaining 100 servers. What is the probability that they all fail on the same day?

That at least one fails today?

② What is the probability that a randomized algorithm returns the wrong answer 10 times in a row?

③ Users are randomly assigned usernames. What is the probability two users are assigned the same username?

Given events $A_1, A_2, \ldots, A_n$, how do we calculate $P(A_1 \cap A_2 \cap \ldots A_n)$ and $P(A_1 \cup A_2 \cup \ldots A_n)$?

↳ prob. that all of $A_1, \ldots, A_n$ occur

↳ prob. that at least 1 of $A_1, \ldots, A_n$ occurs

Three methods:

1. Exact formulas for $P(A_1 \cap \ldots \cap A_n)$ and $P(A_1 \cup \ldots \cup A_n)$

2. Make assumptions about $A_1, \ldots, A_n$ that simplify things

3. Estimate the probabilities, rather than computing them directly

5 Intersections of Events

5.1 Intersections of Events: Exact Formula

Thm (Product rule) If $A_1, A_2, \dots, A_n$ are events s.t. $P(\bigcap_{i \le n} A_i) > 0$ then $P(A_1 \cap A_2 \cap \dots \cap A_n) =$

$P(A \cap B) =$

$P(A \cap B \cap C) =$

pf Induction on n.

Base case:

Inductive step:

$P(A_1 \cap \dots \cap A_n \cap A_{n+1}) =$

Thm (Product rule) If $A_1, A_2, \dots, A_n$ are events s.t. $P(\bigcap_{i \le n} A_i) > 0$
then $P(A_1 \cap A_2 \cap \dots \cap A_n) =$

$$P(A_1)P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2) \cdots P(A_n | A_1 \cap \dots \cap A_{n-1})$$

Example Draw 5 cards from a deck without replacement. What is the probability of getting all hearts?

$A_1 =$

$A_2 =$

$\vdots$

$A_5 =$

$P(\text{all hearts}) =$

5.2 Intersections of Events: Easy Special Cases

1. Mutually independent events

$P(A_1 \cap \ldots \cap A_n) =$

2. Mutually exclusive events

$P(A_1 \cap A_2 \cap \ldots \cap A_n) =$

In the real world, whether or not events are independent is an empirical question

Assuming events are independent can make a **HUGE** difference to probabilities...

100 servers, each one crashes w/ probability $\frac{1}{10}$

Probability all servers crash?

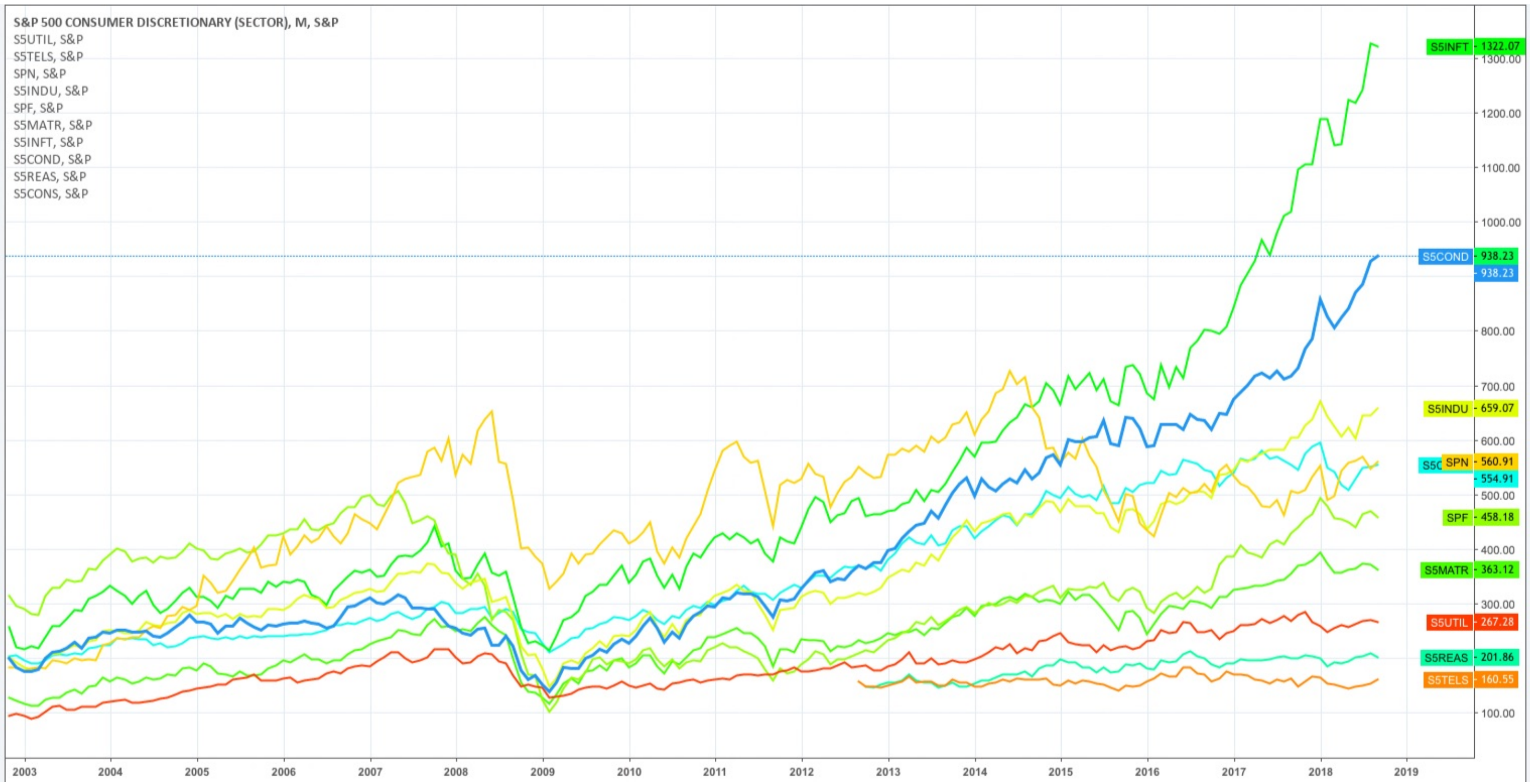
Probability if crashes are independent?

But it can also be dangerous.

S&P 500 tracks 11 categories of stocks in the US stock market
Chance that a single category falls during 1 year:

Assuming independence, chance that all 11 fall in 1 year:

Reality:



S&P 500 CONSUMER DISCRETIONARY (SECTOR), M, S&P
S5UTIL, S&P
S5TELS, S&P
SPN, S&P
S5INDU, S&P
SPF, S&P
S5MATR, S&P
S5INFT, S&P
S5COND, S&P
S5REAS, S&P
S5CONS, S&P
S5INFT 1322.07
S5COND 938.23
938.23
S5INDU 659.07
SPN 560.91
554.91
SPF 458.18
S5MATR 363.12
S5UTIL 267.28
S5REAS 201.86
S5TELS 160.55
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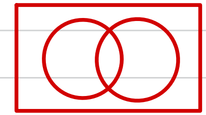
6 Unions of Events

6.1 Unions of Events: Exact Formula



Question What is $P(A_1 \cup A_2)$?

Answer $P(A_1 \cup A_2) =$

Thm (Inclusion-Exclusion, probability version) For any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) =$$

But:

6.2 Unions of Events: Easy Special Cases

1. Mutually independent events

$P(A_1 \cup A_2 \cup \ldots \cup A_n) =$

2. Mutually exclusive events

$P(A_1 \cup A_2 \cup A_3) =$

More generally:

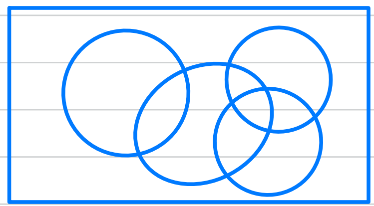
$P(A_1 \cup A_2 \cup \ldots \cup A_n) =$

6.3 Unions of Events: Estimates

Instead of computing $P(A_1 \cup \dots \cup A_n)$ exactly, sometimes it's enough to find an upper bound

Grand champion of all upper bound techniques in probability:

Prop (Union Bound)



⑦ Examples

① 10 termite traps around your house
Each day you randomly choose one to inspect

What is the probability that after 50 days
there is some trap you have not inspected?

(2) Roll a fair die 3 times. Win if you get at least 1 six.

What is your chance of winning?

Method 1:

Method 2: