0 Announcements

Midterm scores are out!
Good job. We're happy with your performance.

Regrade requests: 8 pm Wednesday – midnight Sunday
Estimated grade bins on piazza

Midsemester survey
Please let us know how things are going.
Don't be afraid to be critical!

One-on-one meetings
If you want to talk to a TA about anything related to this class, fill out the form by Wednesday night.

A comment.
Exams and grades are ephemeral. Understanding is what matters.

A reminder:

Monday HW session $\longrightarrow$ Tuesday 10 am – 12 pm (permanently)

HW problems take time! Why not start early?

You can sign up for a small discussion section at any time during the summer.

0.5 Recap

Previously on CS70...

Conditional probability $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

probability that A occurs given that you know B occurred

Bayes' rule $P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}$

prior probability

update

event we are interested in

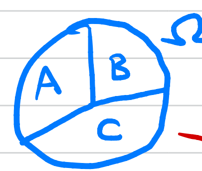
new evidence

Simple proof. Profound implications?

P(You are sick | you test positive)

Base rates matter

Law of total probability



$P(A) + P(B) + P(C) = 1$

disjoint union is Ω

① Independence

Def Events A and B are independent if $P(A \cap B) = P(A)P(B)$

Intuitive meaning: Knowing whether or not A occurred tells you nothing about whether B occurred & vice-versa

Formal version of intuition

Prop If $P(B) > 0$ then A & B are independent if and only if $P(A|B) = P(A)$.

pf ($\Rightarrow$) $P(A \cap B) = P(A)P(B) \Rightarrow P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$

$= P(A)$

($\Leftarrow$) $P(A|B) = P(A) \Rightarrow \dfrac{P(A \cap B)}{P(B)} = P(A)$

$\Rightarrow P(A \cap B) = P(A)P(B)$

Cor If $P(A), P(B) > 0$ then $P(A|B) = P(A) \Leftrightarrow P(B|A) = P(B)$

Def Events A and B are **independent** if $P(A \cap B) = P(A)P(B)$

Intuitive meaning: Knowing whether or not A occurred tells you nothing about whether B occurred

Example Flip a fair coin and then flip it again

A = 1st flip is H
B = 2nd flip is H
} independent (or are they?)
↳ empirical question

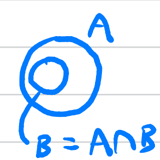
$P(A) = \frac{1}{2}$ $\quad P(B) = \frac{1}{2}$ $\quad P(A \cap B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Independence is often an assumption you make when deciding how to model a real world situation

Example Flip a fair coin 2 times

A = 1st flip is H
B = both flips are H
} not independent

$P(A) = \frac{1}{2}$ $\quad P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\quad P(A \cap B) = \frac{1}{4}$



Question Suppose A and B are independent. What about A and $\overline{B}$?

Answer Also independent! Same for $\overline{A}, B$ and $\overline{A}, \overline{B}$

Intuitively If knowing A doesn't tell you whether B occurred, it also doesn't tell you whether B <u>didn't</u> occur

Prop If A and B are independent events then so are A and $\overline{B}$.

pf

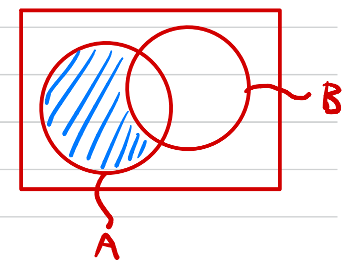
$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

↳ law of total probability

$$= P(A) - P(A)P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A)P(\overline{B})$$



② Pairwise vs. Mutual Independence

Question Suppose A, B, C are events

A, B independent

A, C independent

B, C independent

What is $P(A \cap B \cap C)$?

Answer Trick question! It depends on how $A, B,$ and C are related.

It is not always true that $P(A \cap B \cap C) = P(A)P(B)P(C)$

Example Flip a fair coin 2 times

A = 1st flip is H

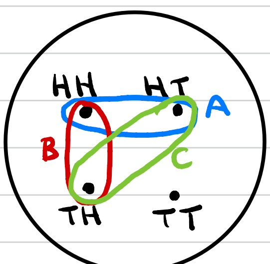
B = 2nd flip is H

C = The two flips are different

$P(A) = 1/2$

$P(B) = 1/2$

$P(C) = 1/2$



$A \cap B = HH \qquad P(A \cap B) = 1/4$

$A \cap C = HT \qquad P(A \cap C) = 1/4$

$B \cap C = TH \qquad P(B \cap C) = 1/4$

$$\boxed{P(A \cap B \cap C) = 0}$$

Def Events $A_1, A_2, \dots, A_n$ are pairwise independent if for all $i \neq j$, A_i and A_j are independent

Def Events $A_1, A_2, \dots, A_n$ are mutually independent if for all $I \subseteq \{1, 2, 3, \dots, n\}$ s.t. $|I| \geq 2$,

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

What does this mean for $n = 3$?

$$P(A_1 \cap A_2) = P(A_1) P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2) P(A_3)$$

$\rightarrow$ pairwise independence

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$$

Note: Mutual independence $\Rightarrow$ Pairwise independence

Pairwise independence $\not\Rightarrow$ Mutual independence

A familiar example...

Suppose you want to share a secret $s \in \{0, 1, 2, \ldots, 10\}$

Let's assume $s = 7$

Generate a polynomial $p(x) = a_3 x^3 + a_2 x^2 + a_1 x + s$ over $GF(11)$
by picking a_3, a_2, a_1 randomly from $\{0, 1, 2, \ldots, 10\}$

Hand out shares $p(x_0), p(x_1), \ldots, p(x_n)$ $\quad x_i \neq x_j \quad x_i \neq 0$

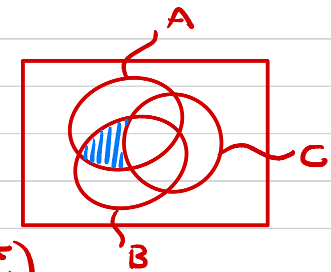
Define A_i = event that $p(x_i) = 0$

<u>Question</u> For what n are A_i's mutually independent?

<u>Answer</u> For $n < 4$, mutually independent

For $n \geq 4$, not mutually independent $\rightarrow$ $P(A_1 \cap A_2 \cap A_3 \cap A_4) = 0$
but pairwise independent
(actually 3-wise independent)

Question Suppose A, B, C mutually independent. What is $P(A \cap B \cap \overline{C})$?



Answer $P(A \cap B \cap \overline{C}) = P(A)P(B)(1 - P(C))$

Why? $P(A \cap B \cap \overline{C}) = P(A \cap B) - P(A \cap B \cap C)$

$$A \cap B = (A \cap B \cap C) \cup (A \cap B \cap \overline{C})$$

$$= P(A)P(B) - P(A)P(B)P(C)$$

$$= P(A)P(B)(1 - P(C))$$

Prop $A_1, A_2, \ldots, A_n$ are mutually independent if and only if for all

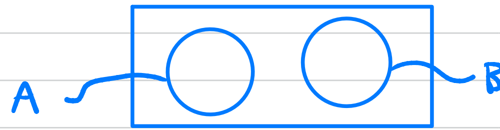
$$B_1 \in \{A_1, \overline{A_1}\}$$
$$B_2 \in \{A_2, \overline{A_2}\}$$
$$\vdots$$
$$B_n \in \{A_n, \overline{A_n}\}$$

we have

$$P(B_1 \cap B_2 \cap \cdots \cap B_n) = P(B_1)P(B_2) \cdots P(B_n)$$

③ Mutually Exclusive Events

Def Events A and B are **mutually exclusive** if $A \cap B = \emptyset$



A, B mutually exclusive $\Rightarrow$ $P(A \cap B) = 0$
$P(A \cup B) = P(A) + P(B)$

Example Flip a fair coin 2 times
A = 1^{st} flip is H
B = both flips are T

Question Suppose A, B are independent events with $P(A), P(B) > 0$. Can A & B be mutually exclusive?

Answer No! $P(A \cap B) = P(A)P(B) > 0$

"Independent" and "mutually exclusive" are (almost) opposites

Warning: Mutually exclusive events are (almost) never independent!!

④ Combinations of Events

Given events $A_1, A_2, \ldots, A_n$, how do we calculate $P(A_1 \cap A_2 \cap \ldots A_n)$ and $P(A_1 \cup A_2 \cup \ldots A_n)$?

↳ prob. that all of $A_1, \ldots, A_n$ occur

↳ prob. that at least 1 of $A_1, \ldots, A_n$ occurs

Examples

① You are in charge of maintaining 100 servers. What is the probability that they all fail on the same day?

↳ intersection of events

That at least one fails today? → union of events

② What is the probability that a randomized algorithm returns the wrong answer 10 times in a row?

↳ intersection

③ Users are randomly assigned usernames. What is the probability two users are assigned the same username?

↳ union of intersections $P((A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \ldots)$

Given events $A_1, A_2, \ldots, A_n$, how do we calculate $P(A_1 \cap A_2 \cap \ldots A_n)$ and $P(A_1 \cup A_2 \cup \ldots A_n)$?

↳ prob. that all of $A_1, \ldots, A_n$ occur

↳ prob. that at least 1 of $A_1, \ldots, A_n$ occurs

Three methods:

1. Exact formulas for $P(A_1 \cap \ldots \cap A_n)$ and $P(A_1 \cup \ldots \cup A_n)$
   But often hard to compute
2. Make assumptions about $A_1, \ldots, A_n$ that simplify things
   But these assumptions may be unrealistic
3. Estimate the probabilities, rather than computing them directly
   Sometimes an upper bound is good enough
   Later in this class: 2 more lectures on ways to do this

5 Intersections of Events

5.1 Intersections of Events: Exact Formula

<u>Thm</u> (Product rule) If $A_1, A_2, \dots, A_n$ are events s.t. $P(\bigcap_{i \le n} A_i) > 0$
then $P(A_1 \cap A_2 \cap \dots \cap A_n) =$

$$P(A_1)P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \dots \cap A_{n-1})$$

$P(A \cap B) = P(A)P(B|A)$

$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$

pf Induction on n.

Base case: $P(A_2|A_1) = \dfrac{P(A_2 \cap A_1)}{P(A_1)}$

$$\Rightarrow P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$$

Inductive step:

by $n=2$ case

$$P(\underline{A_1 \cap \dots \cap A_n} \cap \underline{A_{n+1}}) = P(A_1 \cap \dots \cap A_n)\,P(A_{n+1}|A_1 \cap \dots \cap A_n)$$

IH

$$= P(A_1)P(A_2|A_1) \cdots P(A_{n+1}|A_1 \cap \dots \cap A_n)$$

Thm (Product rule) If $A_1, A_2, \ldots, A_n$ are events s.t. $P(\bigcap_{i \le n} A_i) > 0$ then $P(A_1 \cap A_2 \cap \ldots \cap A_n) =$

$$P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \ldots \cap A_{n-1})$$

Example Draw 5 cards from a deck without replacement. What is the probability of getting all hearts?

$A_1 =$ 1^{st} card is hearts
$A_2 =$ 2^{nd} card is hearts
$\vdots$
$A_5 =$ 5^{th} card is hearts

$$P(\text{all hearts}) = P(A_1 \cap A_2 \cap \ldots \cap A_5)$$

$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_5|A_1 \cap \ldots \cap A_4)$$

$$= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = {}_{13}P_5 / {}_{52}P_5$$

5.2 Intersections of Events: Easy Special Cases

1. Mutually independent events

$$P(A_1 \cap \dots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

2. Mutually exclusive events

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = 0$$

In the real world, whether or not events are independent is an empirical question

Assuming events are independent can make a **HUGE** difference to probabilities...

100 servers, each one crashes w/ probability $\frac{1}{10}$

Probability all servers crash? Could be as high as $\frac{1}{10}$

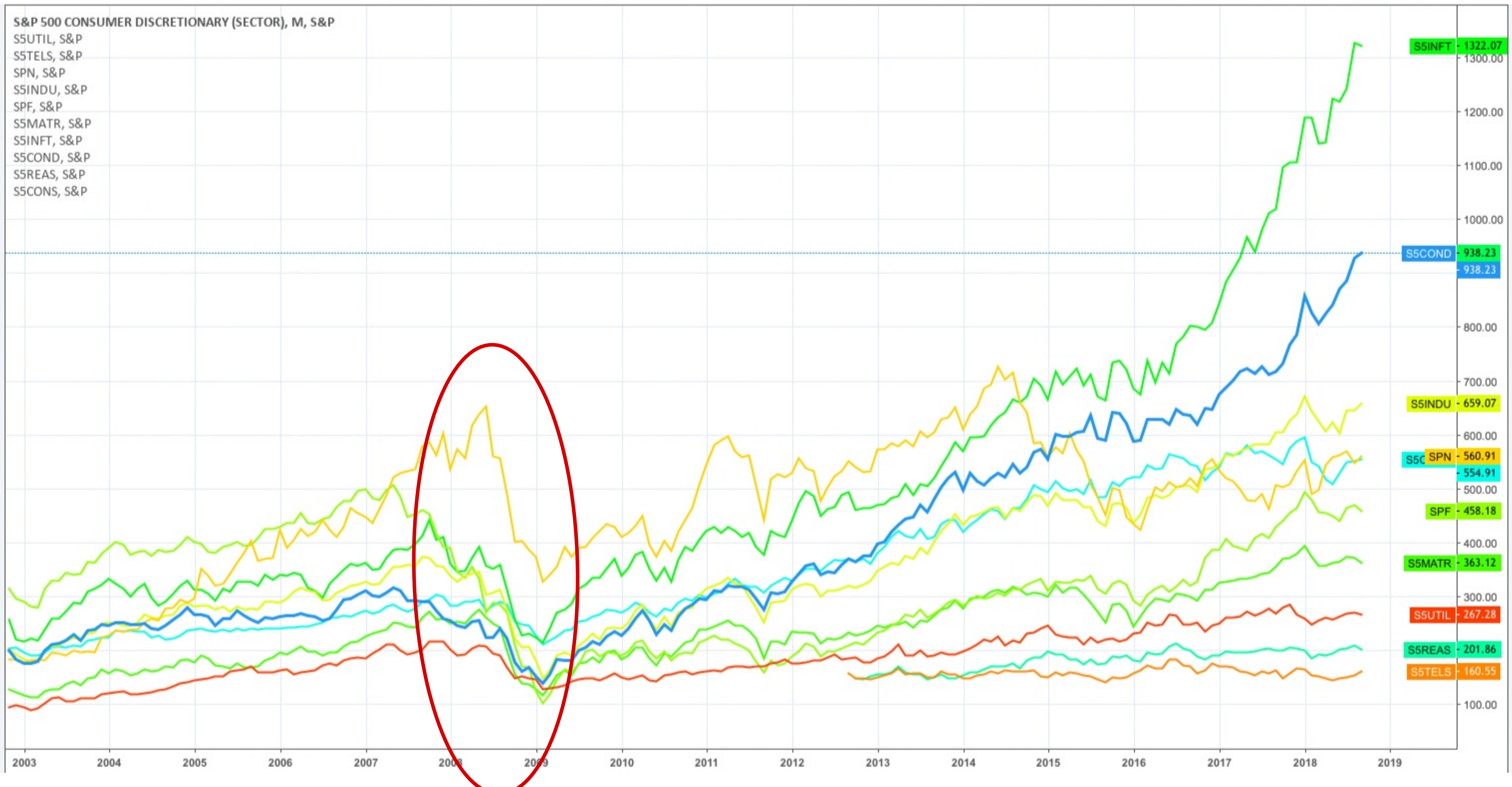
Probability if crashes are independent? $\left(\frac{1}{10}\right)^{100}$
(less than 1/(# of atoms in the observable universe))

But it can also be dangerous.

S&P 500 tracks 11 categories of stocks in the US stock market
Chance that a single category falls during 1 year: ~ 1/4

Assuming independence, chance that all 11 fall in 1 year: ~ $\frac{1}{4\text{ million}}$

Reality: In 2008, all categories fell



S&P 500 CONSUMER DISCRETIONARY (SECTOR), M, S&P
S5UTIL, S&P
S5TELS, S&P
SPN, S&P
S5INDU, S&P
SPF, S&P
S5MATR, S&P
S5INFT, S&P
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S5CONS, S&P
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S5COND 938.23
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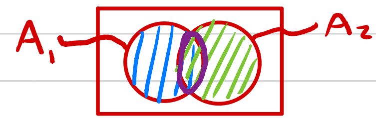
6 Unions of Events

(6.1) Unions of Events: Exact Formula

Question What is $P(A_1 \cup A_2)$?



Answer $P(A_1 \cup A_2) = \underbrace{P(A_1) + P(A_2)}_{\text{"overcounting"}} - P(A_1 \cap A_2)$

Inclusion-Exclusion works for probability

Thm (Inclusion-Exclusion, probability version) For any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{\substack{i \neq j \neq k \\ i \neq k}} P(A_i \cap A_j \cap A_k)$$

$$- \ \ldots \ + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

But: Computing intersections can also be hard
And there are many terms

6.2 Unions of Events: Easy Special Cases

① Mutually independent events

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n)$$

↳ De Morgan's laws

$$= 1 - P(\bar{A}_1)P(\bar{A}_2)\cdots P(\bar{A}_n)$$

↳ independence

$$= 1 - (1 - P(A_1))(1 - P(A_2))\cdots(1 - P(A_n))$$

② Mutually exclusive events

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - \cancel{P(A_1 \cap A_2)}^{0} - \cancel{P(A_1 \cap A_3)}^{0} - \cancel{P(A_2 \cap A_3)}^{0} + \cancel{P(A_1 \cap A_2 \cap A_3)}^{0}$$

More generally:

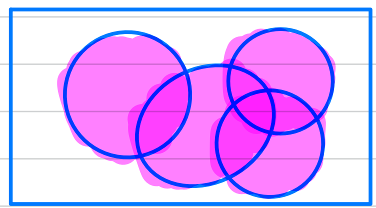
$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$$

6.3 Unions of Events: Estimates

Instead of computing $P(A_1 \cup \ldots \cup A_n)$ exactly, sometimes it's enough to find an upper bound

Grand champion of all upper bound techniques in probability:

Prop (Union Bound) $P(A_1 \cup \ldots \cup A_n) \le P(A_1) + \ldots + P(A_n)$



→ overlapping areas get counted multiple times

Simple, but surprisingly useful

⑦ Examples

① 10 termite traps around your house
Each day you randomly choose one to inspect

What is the probability that after 50 days there is some trap you have not inspected?

A_1 = never inspect 1^{st} trap
⋮
A_{10} = never inspect 10^{th} trap

Not mutually exclusive (or independent)

Want: $P(A_1 \cup A_2 \cup \ldots \cup A_{10})$

$P(A_i) = \left(\frac{9}{10}\right)\cdot\left(\frac{9}{10}\right)\cdot \ldots \cdot\left(\frac{9}{10}\right) = \left(\frac{9}{10}\right)^{50}$

↪ days are independent

Union Bound: $P(A_1 \cup \ldots \cup A_{10}) \leq P(A_1) + \ldots + P(A_{10})$
$= 10 \cdot \left(\frac{9}{10}\right)^{50}$
≈ 0.05 ← Good enough?

② Roll a fair die 3 times. Win if you get at least 1 six.

What is your chance of winning?

Method 1: inclusion - exclusion

A_i = get a six on the i^{th} roll.

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\
&\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&\quad + P(A_1 \cap A_2 \cap A_3) \\
&= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^3
\end{aligned}
$$

Method 2: independence

Note A_1, A_2, A_3 are independent

$$P(A_1 \cup A_2 \cup A_3) = 1 - (1 - P(A_1))(1 - P(A_2))(1 - P(A_3)) = 1 - \left(\frac{5}{6}\right)^3$$