## (-1) A Correction

Regrades open [redacted], not Monday

## ⓪ More tips on problem solving

Every challenging mathematical problem is its own little world.

Some strategies

① Set the parameters to extreme values and see what happens.

② Try out small examples where you can brute force the calculations.

③ Make a guess about what's going on &

## 0.5 Recap

Previously on CS 70...

1. Independent events $P(A \cap B) =$

2. Intersections of events $P(A \cap B \cap C) =$

   Easy if A, B, C

3. Unions of events $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

   Union bound: $P(A \cup B \cup C)$

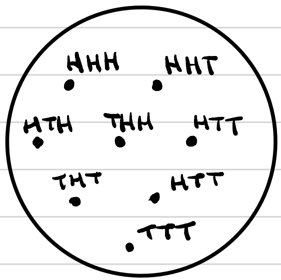
When you can't calculate exactly,

## ① Random variables

Common situation: want to measure some quantity of the outcomes in a sample space and answer questions like:

- 
- 
- 

**Example** Flip a fair coin 3 times & count the number of heads




Possible values | Probability

**Def** Suppose $\Omega$ is a sample space. A random variable
is

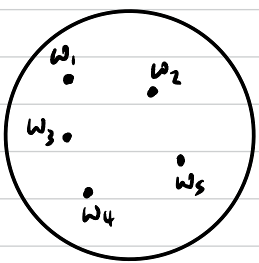
$P(X = a)$ =

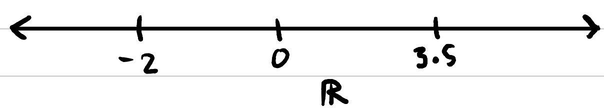
The distribution of $X$ is

that tells us for each possible value of $X$,

Distribution of $X$

$\Omega$



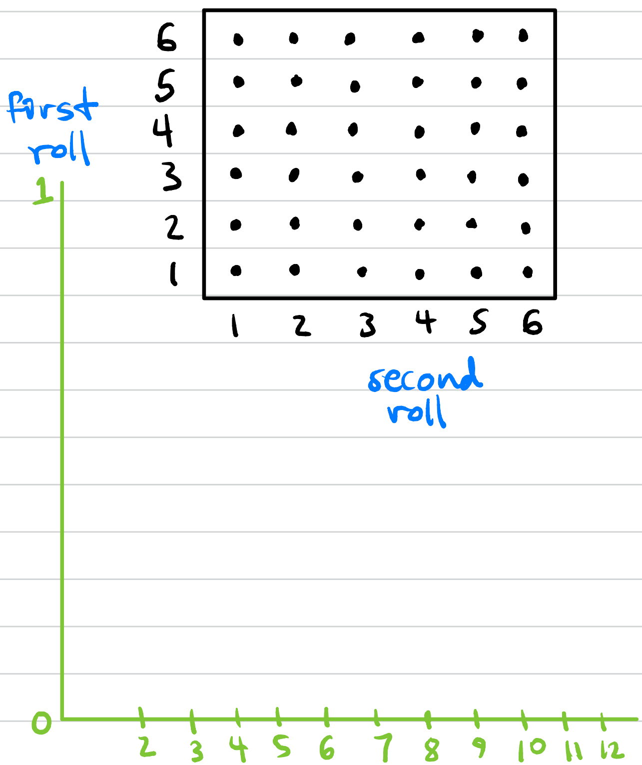



**Example** Roll two fair dice and let $X =$

$X((1,4)) =$

$X((5,2)) =$

**Distribution**



## ② Random variables with fancy names

Some types of random variables show up so often that we give them fancy names

Actually, technically we name

## (2.1) Bernoulli Distribution

Have a biased coin which is heads with probability p.
Flip it once. Define a random variable X by

Distribution of X

This is called the and
we say

Fancy name, simple concept

Comment: How to get a simple concept in math named after you?

**Example** Roll a fair 6-sided die. Define a random variable $X$ by

**Question**

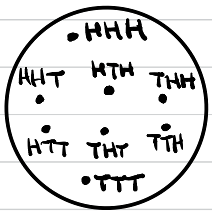
**Answer**

**The point**

## 2.2 Binomial distribution

Have a biased coin which is heads with probability $p$.
Flip it $n$ times. Let $X =$

Example $n=3$, $p=1/3$




$P(X=2) =$

Distribution of $X$

This is called the

Written

__Example__ When data is sent over the internet, it is broken up into small chunks called packets.

Each packet is sent separately and usually a few of the packets fail to reach the intended destination

This is often modelled as

What assumptions are we making?

## ③ Joint distributions

<u>Def</u> Suppose $\Omega$ is a sample space and $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ are two random variables. The joint distribution of $X$ and $Y$ is

<u>Example</u> Flip a fair coin 3 times.

$X = \#$ of heads

$$Y = \begin{cases} 1 & \text{if } 1^{st} \text{ flip is } H \\ 0 & \text{if } 1^{st} \text{ flip is } T \end{cases}$$

| $\omega$ | $X(\omega)$ | $Y(\omega)$ |
|---|---|---|
| HHH | | |
| HHT | | |
| HTH | | |
| THH | | |
| HTT | | |
| THT | | |
| TTH | | |
| TTT | | |

<u>Joint distribution</u>

## 3.1 Marginal distributions

Suppose we have 2 random variables and we know their joint distribution. How do we

Example Suppose we have random variables X and Y

Joint distribution of X & Y:

| $X \backslash Y$ | 1 | 2 | 3 |
|---|---|---|---|
| -2 | 0 | 1/12 | 2/12 |
| 0 | 1/12 | 1/12 | 3/12 |
| 1.5 | 1/12 | 0 | 0 |
| 3 | 0 | 1/12 | 2/12 |

Question

Answer

Suppose we have 2 random variables and we know their joint distribution. How do we find the distribution of $X$ by itself?

"Marginal distribution for $X$"

**Prop** For any random variables $X$ and $Y$,

$$P(X=a) =$$

pf

## 3.2 Independence of random variables

**Question** Given distribution of X and distribution of Y, how can you compute the joint distribution of X and Y?

**Answer**

**Def** Random variables X and Y are independent if

Intuitive idea:

**Example** Roll 2 fair dice

$X$ = value of $1^{st}$ roll

$Y$ = value of $2^{nd}$ roll

$Z$ = sum of the two rolls

**Question** $X$ and $Y$ independent?

**Answer**

**Question** $X$ and $Z$ independent?

**Answer**

When you have > 2 random variables

Just like events,

**Def** Random variables $X_1, X_2, \ldots, X_n$ are pairwise independent if

**Def** Random variables $X_1, X_2, \ldots, X_n$ are mutually independent if

Common abbreviation means $X_1, X_2, \ldots, X_n$ are i.i.d. random variables

Note: "Independent" usually means

## 3.3 Combining random variables

Given random variables $X, Y$ it is common to form a new random variable

**Example** Flip a fair coin 3 times

$X =$

$Y =$

Define $Z =$

| $\omega$ | $X(\omega)$ | $Y(\omega)$ | $Z(\omega)$ |
|---|---|---|---|
| HHH | 3 | 1 | |
| HHT | 2 | 1 | |
| HTH | 2 | 1 | |
| THH | 2 | 0 | |
| HTT | 1 | 1 | |
| THT | 1 | 0 | |
| TTH | 1 | 0 | |
| TTT | 0 | 0 | |

Question

Answer

## ④ Expected value

**Question** I flip a fair coin 3 times and pay you \$1 for each H. How much would you pay me to play this game?

**Answer**

**Def** If $X$ is a random variable, the expected value of $X$, written $E(X)$, is defined by

$$E(X) =$$

**Note**:

If you do something with expected value $a$ and you repeat it many times (independently) then in the long run the average of those times will be close to $a$

**Def** If $X$ is a random variable, the expected value of $X$, written $E(X)$, is defined by

$$E(X) = \sum_{a \in \text{range}(X)} a \cdot P(X=a)$$

**Example** Flip a coin 3 times. $X =$

$E(X) =$

**Prop** $E(X) = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$

~~pf~~

<u>Example</u> $X \sim \mathrm{Bin}(n, p)$

$E(X) =$

Enter, the hero:

## 5. Linearity of Expectation

Thm For any random variables $X_1, X_2, \ldots, X_n$

$$E(X_1 + X_2 + \ldots + X_n) =$$

pf.

# Expected value of binomial distribution, revisited

Have a coin which is heads with probability $p$
Flip it $n$ times
$X$ = # of heads

**Cor**