## ⓪ Comment on yesterday's lecture

Pace was too fast at the end of lecture

I went quickly to try to finish everything on time

Today:

I know some people like these things, but time is finite & going at an appropriate pace is more important

Start of lecture today:

## ① Combining random variables

$X, Y$ random variables
New random variables by combining $X$ and $Y$:

Example Flip a fair coin 3 times

$X = \#$ of heads

$$Y = \begin{cases} 1 & \text{if } 1^{st} \text{ flip is heads} \\ 0 & \text{if } 1^{st} \text{ flip is tails} \end{cases}$$

$Z =$

Question What is $Z(HHT)$?

Answer

## ② Expected value

$X : \Omega \to \mathbb{R}$ random variable

Definition:

Equivalent Formula:

Example Flip a coin 2 times

$X =$

sum over possible values: $E(X) =$

sum over outcomes: $E(X) =$

## (2.1) Useful Facts

$X, Y$ random variables
$a \in \mathbb{R}$

**Prop** $E(aX) =$

**Prop** $E(a) =$

**Prop** For any function $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$E(f(X, Y)) =$

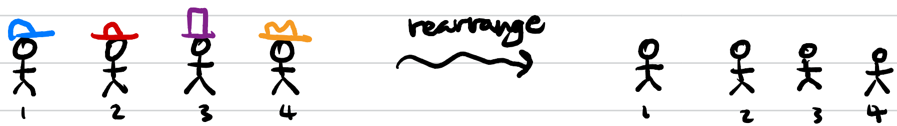
**Example** $X$ and $Y$ both have range $\{0, 1, 2, \dots, n\}$

$E(XY) =$

## (2.2) Linearity of Expectation

**Thm** $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$

**Example** n people, each wearing a hat
collect all hats & reassign them randomly




$X =$

What is $E(X)$?

**Attempt 1** Possible values of $X$:

$E(X) =$

**Thm** $E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$

**Example** n people, each wearing a hat
collect all hats & reassign them randomly

$X$ = # of students who get their own hat back

What is $E(X)$?

**Attempt 2** Define random variables

$X_1 =$

$\vdots$

$X_n =$

$E(X_i) =$

$E(X) =$

## ③ Variance

$X$ is a random variable

**Def** If $X$ is a random variable with expected value $E(X) = a$ then the variance of $X$, written $\mathrm{Var}(X)$ is

$$\mathrm{Var}(X) =$$

**Comment** Standard deviation of $X$:

**Def** If $X$ is a random variable with expected value $E(X) = a$ then the variance of $X$, written $Var(X)$ is

$$Var(X) = E\left((X-a)^2\right)$$

**Example** Roll a fair 6-sided die. $X =$

$E(X) =$

| outcome | $X$ | $(X-3.5)^2$ |
| --- | --- | --- |
| 1 | 1 | |
| 2 | 2 | |
| 3 | 3 | |
| 4 | 4 | |
| 5 | 5 | |
| 6 | 6 | |

$Var(X) = E\left((X-3.5)^2\right) =$

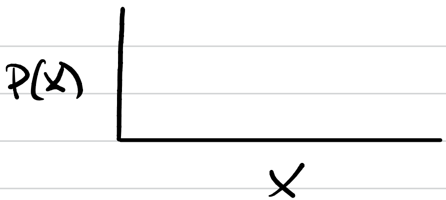
## (3.1) Meaning of variance

$a = E(X)$

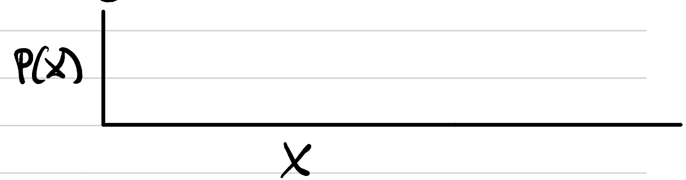
$Var(X) = E((X-a)^2)$

Intuitively:

low variance:




high variance:




**Question** Why not $E(X-a)$?

**Answer**

**Question** Why not $E(|X-a|)$?

**Answer**

**Question** Why not $E(X-a)$?

**Answer** Positive and negative deviations cancel out

**Example** Roll a fair 6-sided die

$X$ = value of roll

$E(X) = 3.5$

| outcome | $X$ | $X - 3.5$ |
|---|---|---|
| 1 | 1 | |
| 2 | 2 | |
| 3 | 3 | |
| 4 | 4 | |
| 5 | 5 | |
| 6 | 6 | |

$E(X - 3.5) =$

## (3.2) Alternative formula for variance

Thm For any random variable $X$,

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

pf

**Thm** For any random variable $X$, $Var(X) = E(X^2) - (E(X))^2$

**Example** (A bit tricky) $X \sim Bin(n,p)$
What is $Var(X)$?

$Var(X) =$

**Final answer**

## ④ Covariance

Random variables $X$ and $Y$

**Def** If $X$ is a random variable with $E(X) = a$ and $Y$ is a random variable with $E(Y) = b$ then the covariance of $X$ and $Y$, written $Cov(X, Y)$ is

**Question** If you just know the distribution of $X$ and the distribution of $Y$, can you calculate $Cov(X, Y)$?

**Answer**

**Def** $E(X) = a \quad E(Y) = b \quad \Rightarrow \quad Cov(X, Y) = E((X-a)(Y-b))$

**Example** Flip a fair coin 3 times

$X =$

$Y =$

$Cov(X, Y) =$

| Outcome | X | Y | X-1.5 | Y-1.5 | (X-1.5)(Y-1.5) |
|---|---|---|---|---|---|
| HHH | | | | | |
| HHT | | | | | |
| HTH | | | | | |
| THH | | | | | |
| HTT | | | | | |
| THT | | | | | |
| TTH | | | | | |
| TTT | | | | | |

## 4.1 Meaning of covariance

Sign of $Cov(X, Y)$

$\underline{Cov(X,Y) < 0}$ $\underline{Cov(X,Y) = 0}$ $\underline{Cov(X,Y) > 0}$

Magnitude of $Cov(X, Y)$

More meaningfull measure:

Examples What is the sign of $\mathrm{Cov}(X,Y)$?

① Pick a random day from 2020
$X$ = max temperature in Berkeley that day
$Y$ = max temperature in Berkeley the next day

② Pick a random day from 2020
$X$ = max temperature in Berkeley that day
$Y$ = number of people in Berkeley who used their heater that day

③ Pick a random day from 2020
$X$ = max temperature in Berkeley that day
$Y$ = what day of the month it was

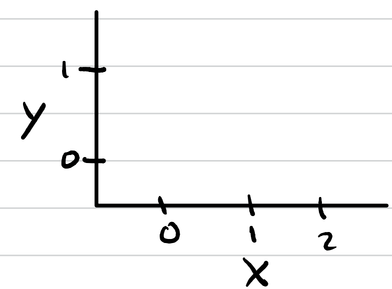
**Warning: $Cov(X,Y)=0$ does not mean $X, Y$ independent!**

Example: Flip a fair coin 2 times

$X =$ $\qquad\qquad\qquad$ $E(X)=$

$Y =$ $\qquad\qquad\qquad$ $E(Y)=$

| Outcome | $X$ | $Y$ | $X-1$ | $Y-1/2$ | $(X-1)(Y-1/2)$ |
|---|---|---|---|---|---|
| HH | | | | | |
| HT | | | | | |
| TH | | | | | |
| TT | | | | | |




$Cov(X,Y) =$

$X, Y$ not independent, but

## ⑤ Facts about $E(X)$, $Var(X)$, $Cov(X,Y)$

Properties are useful, the proofs are not that important

① Covariance is symmetric

$Cov(Y, X) =$

② Covariance is bilinear

$Cov(X+Y, Z) =$

$Cov(a \cdot X, Y) =$

$Cov(a_1 X_1 + a_2 X_2 + \dots + a_n X_n, Y) =$

3 Variance in terms of covariance

$Var(X) =$

4 Covariance in terms of expected value

$Cov(X, Y) =$

5 Variance of a sum

$Var(X + Y) =$

## 5.1 When $X, Y$ are independent

Suppose $X$ and $Y$ are independent

① $E(XY) = E(X)E(Y)$

② $Cov(X, Y) = 0$

③ $Var(X+Y) = Var(X) + Var(Y)$