Announcements

Regrade requests

If you lost points because you selected the wrong pages for a problem, you should submit a regrade request

Midsemester survey

Please fill it out if you can

Goal for today

Two more common distributions

Derive their basic properties

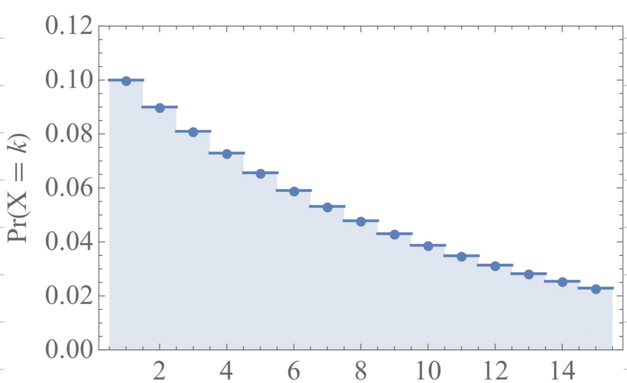
Why? Want to build probabilistic models of situations in science or technology

① <u>Geometric distribution</u>

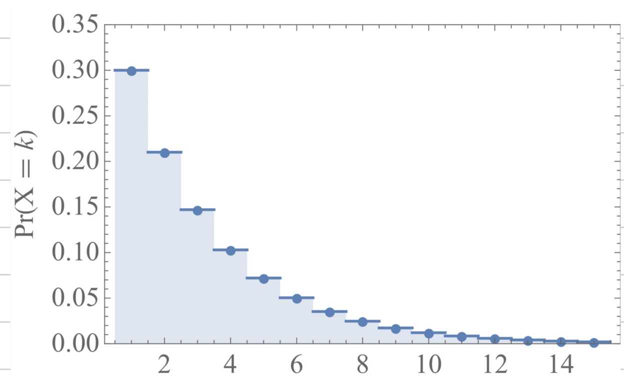
Have a biased coin which is heads with probability p
Flip it until you get heads

X =

Example



p = 0.1



p = 0.3

Have a biased coin which is heads with probability p
Flip it until you get heads

X = # of flips

Example

TTTH $X = 4$ probability:

TTTTTTH $X = 7$ probability:

H $X = 1$ probability:

Distribution of X Possible values:

$P(X = i) =$

This distribution is called the ________ and we write ________

Reminder $X \sim \text{Geometric}(p) \Rightarrow P(X=i) = (1-p)^{i-1}p$

Question What is $\sum_{i=1}^{\infty} P(X=i)$?

Answer

$$\sum_{i=1}^{\infty} P(X=i) =$$

Lesson

(1.1) Expected value of geometric distribution

Thm If $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1}{p}$.

pf $E(X) =$

$E(X) \quad =$

Warning Can only manipulate infinite series like this when they are

Thm If $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1}{p}$.

Example I roll a fair 6-sided die until I get a 6. If it takes n rolls,

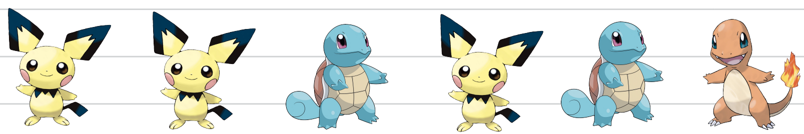
Question How much are you willing to pay to play this game with me?

Answer

1.2 Example: Coupon Collector's Problem

n types of Pokémon 

Each day you catch a random Pokémon



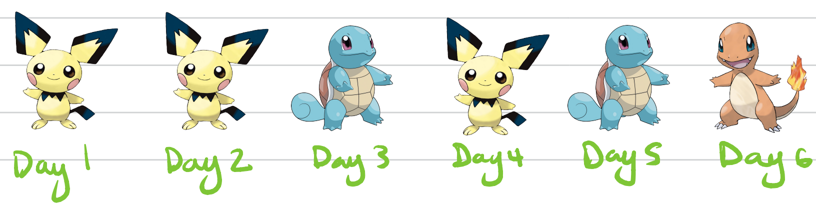
Define $X =$

Define X = # of days until you have one of each type
What is $E(X)$?

Define new random variables

Note:

Example



X = # of days until we get all n types

For $i = 1, 2, \ldots, n$ $\quad X_i$ = # of days after we have $i-1$ types until we have i types

Note: $X = X_1 + X_2 + \ldots + X_n$

$E(X) =$

Upshot:

What is $E(X_i)$?

Distribution of X_i:

$\Rightarrow E(X_i) =$

(1.3) Variance of geometric distribution

Thm If $X \sim \text{Geometric}(p)$ then $\text{Var}(X) = \frac{1-p}{p^2}$

pf $\text{Var}(X) =$

$E(X^2) =$

$\Rightarrow$

Thm If $X \sim \text{Geometric}(p)$ then $\text{Var}(X) = \frac{1-p}{p^2}$

pf (continued) $\text{Var}(X) = E(X^2) - E(X)^2 =$

Have: $E(X^2) - (1-p)E(X^2) = \sum_{i=1}^{\infty} (2i-1)(1-p)^{i-1} p$

$=$

$\Rightarrow$

Comment

② Poisson distribution

Poisson $\neq$ Poison

Def A random variable X has the Poisson distribution with parameter λ if the distribution of X is

we write $X \sim \text{Poisson}(\lambda)$ to indicate this

Comment Bernoulli

Binomial

Geometric

Poisson

2.1 Intuitive meaning of Poisson distribution

Poisson distribution: $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$

Intuitive meaning

Some event occurs

# of times it occurs in any 2 disjoint intervals are

$X =$

What random variables have Poisson distribution?

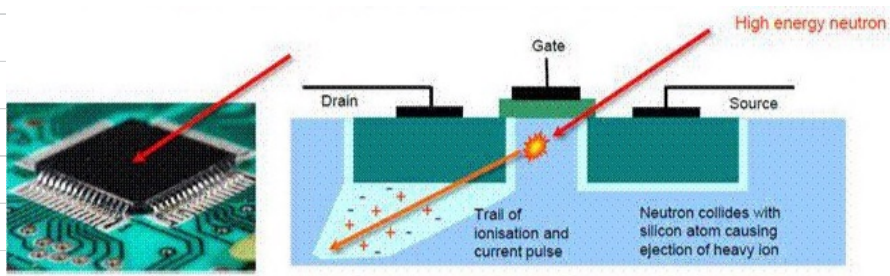
1.

2.

A cool example

Sometimes, cosmic rays

When they do, they can cause



This is a real problem!

How many bits do we expect to flip this way per year?

Poisson distribution: $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$

<u>Uses of the Poisson distribution</u>

Use #1: Model various naturally occurring situations in science, technology, etc

Use #2: Approximation to the binomial distribution

One formalization of this:

<u>Thm</u>

2.2 Calculating things about the Poisson distribution

Suppose $X \sim \text{Poisson}(\lambda)$

What are $E(X)$ and $Var(X)$?

Warm-up $\sum_{k=0}^{\infty} P(X=k) =$

pf

Comment

Lesson:

<u>Thm</u> If $X \sim \text{Poisson}(\lambda)$ then $E(X) = \lambda$

pf

Consider $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

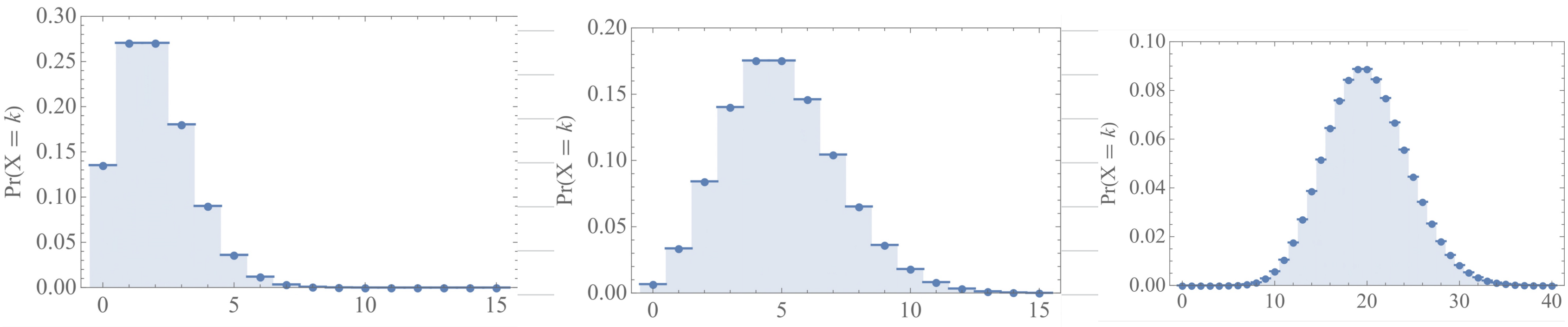
<u>Thm</u> If $X \sim \text{Poisson}(\lambda)$ then $\text{Var}(X) = \lambda$

pf

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

<u>Comment</u> Common way to check if Poisson distribution is a good model for a situation:

What does Poisson(λ) look like?



$\lambda = 3$ $\lambda = 5$ $\lambda = 20$

As $\lambda \to \infty$,

Later, we will see this is explained by the following fact

<u>Thm</u> If $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ) are independent then

<u>Thm</u> If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent then $X+Y \sim \text{Poisson}(\lambda+\mu)$

pf Let $k \in \mathbb{N}$.
Need to show

Thm For $n = 1, 2, 3, \ldots$ let $X_n \sim \mathrm{Bin}\left(n, \frac{\lambda}{n}\right)$

Then for any $k \in \mathbb{N}$

$$\lim_{n \to \infty} P(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Intuition Poisson(λ)

expected number of events:

1

2

3

<u>Thm</u> For $n = 1, 2, 3, \ldots$ let $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$

Then for any $k \in \mathbb{N}$

$$\lim_{n\to\infty} P(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Pf

③ Summary

Bernoulli	$X \sim \text{Bernoulli}(p)$	$E(X) = p$ $Var(X) = p(1-p)$
Binomial	$X \sim \text{Bin}(n, p)$	$E(X) = np$ $Var(X) = np(1-p)$
Geometric	$X \sim \text{Geometric}(p)$	$E(X) = \frac{1}{p}$ $Var(X) = \frac{1-p}{p^2}$
Poisson	$X \sim \text{Poisson}(\lambda)$	$E(X) = \lambda$ $Var(X) = \lambda$