Announcements

Regrade requests

Open until Sunday night

If you lost points because you selected the wrong pages for a problem, you should submit a regrade request

Midsemester survey

Still open

Please fill it out if you can

0 Goal for today

Two more common distributions

somewhat more complicated than Bernoulli or binomial ←

- Geometric distribution
- Poisson distribution

Derive their basic properties

- Expected value
- Variance

Why? Want to build probabilistic models of situations in science or technology

These distributions are often used as building blocks or starting points

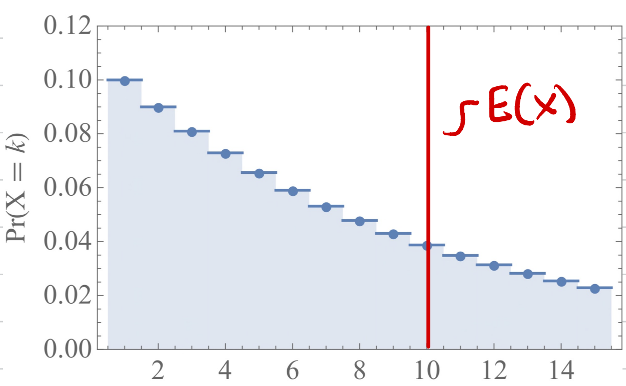
① Geometric distribution

Have a biased coin which is heads with probability p
Flip it until you get heads

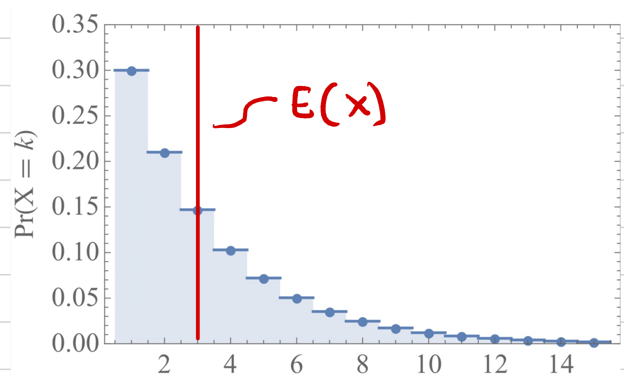
$X = \#$ of flips

Example

TTTH	$X = 4$
TTTTTTH	$X = 7$
H	$X = 1$



$p = 0.1$



$p = 0.3$

Have a biased coin which is heads with probability p
Flip it until you get heads

X = # of flips

<u>Example</u>

TTTH	$X = 4$	probability: $(1-p)(1-p)(1-p)p$
TTTTTTH	$X = 7$	probability: $(1-p)^6 \cdot p$
H	$X = 1$	probability: p

<u>Distribution of X</u> Possible values: 1, 2, 3, 4, 5, ... infinite!
(Aside: What if you never get heads?)

$$P(X = i) = \underbrace{(1-p)^{i-1}}_{\text{fail to get heads on first } i-1 \text{ flips}} \; p$$

p → get heads on ith flip

This distribution is called the geometric distribution with parameter p and we write

$$X \sim \text{Geometric}(p)$$

Reminder $X \sim \text{Geometric}(p) \Rightarrow P(X=i) = (1-p)^{i-1} p$

Question What is $\sum_{i=1}^{\infty} P(X=i)$?

Answer Should be 1 (law of total probability)

Let's check

$$\sum_{i=1}^{\infty} P(X=i) = \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p$$

$$= p \sum_{i=1}^{\infty} (1-p)^{i-1}$$

$$= p \sum_{i=0}^{\infty} (1-p)^{i}$$

Sum of a geometric series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{for } 0<r<1$$

$$= p \left(\frac{1}{1-(1-p)}\right)$$

$$= p \cdot \left(\frac{1}{p}\right) = 1 \checkmark$$

Lesson Working with geometric distributions requires working with infinite sums

(1.1) Expected value of geometric distribution

<u>Thm</u> If $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1}{p}$.

pf $E(X) = \sum_{i=1}^{\infty} i \cdot P(X=i)$

$= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} p$ ← requires a trick to compute (can also use calculus)

$$
\begin{aligned}
E(X) &= 1\cdot p + 2\cdot(1-p)\cdot p + 3\cdot(1-p)^2\cdot p + \dots \\
-(1-p)E(X) &= -0 - 1\cdot(1-p)\cdot p - 2\cdot(1-p)^2\cdot p - \dots \\
\hline
E(X)-(1-p)E(X) &= p + (1-p)p + (1-p)^2\cdot p + \dots \\
&= \sum_{i=1}^{\infty} (1-p)^{i-1}\cdot p = 1
\end{aligned}
$$

→ from prev. slide

$\Rightarrow E(X) - (1-p)E(X) = 1$

$\Rightarrow E(X) = 1/(1-(1-p)) = \frac{1}{p}$

<u>Warning</u> Can only manipulate infinite series like this when they are absolutely convergent. (see math 1A)

Thm If $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1}{p}$.

Example I roll a fair 6-sided die until I get a 6.
If it takes n rolls, I pay you \$$n$

Question How much are you willing to pay to play this game with me?

Answer Let Y = # of rolls
$Y \sim \text{Geometric}(p)$ where $p = \frac{1}{6}$

$E(Y) = 1/(1/6) = 6$

You should pay up to \$6

(1.2) Example: Coupon Collector's Problem

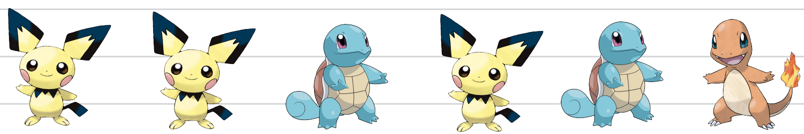
n types of Pokémon 

Each day you catch a random Pokémon

- Each Pokémon equally likely
- Which Pokémon you catch each day are independent

How many days before you catch them all?

↳ i.e. at least 1 of all n types



Day 1 Day 2 Day 3 Day 4 Day 5 Day 6

6 days

Define X = # of days until you have one of each type

What is $E(X)$?

Seems hard.

Linearity of expectation to the rescue!

Define X = # of days until you have one of each type
What is $E(X)$?

Define new random variables
X_1 = # of days to get 1 type of Pokémon
X_2 = # of days after getting 1 type until we get a 2^{nd} type
⋮
X_n = # of days after getting $(n-1)$ types until we get all n types

clever choice

Note: $X = X_1 + X_2 + \ldots + X_n$

Example

$X_1 = 1$
$X_2 = 2$
$X_3 = 3$
$X = 1+2+3 = 6$

X = # of days until we get all n types

For $i = 1, 2, \ldots, n$ $\quad X_i$ = # of days after we have $i-1$ types until we have i types

Note: $X = X_1 + X_2 + \ldots + X_n$

$$E(X) = E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$
$$= \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \ldots + \frac{n}{n-(n-1)}$$
$$= n \sum_{i=1}^{n} \frac{1}{i}$$
$$\approx n(\ln(n) + \gamma)$$

Euler-Mascheroni constant ≈ 0.5772

Upshot: it takes $O(n\log(n))$ days

What is $E(X_i)$?

Have $i-1$ types $\Rightarrow$ chance of getting an i^{th} type each day is $\left(\frac{n-(i-1)}{n}\right)$

# of types you don't yet have

total # of types

Distribution of X_i: Geometric$\left(\frac{n-i+1}{n}\right)$

$$\Rightarrow E(X_i) = \frac{n}{n-i+1}$$

1.3 Variance of geometric distribution

Thm If $X \sim \text{Geometric}(p)$ then $\text{Var}(X) = \frac{1-p}{p^2}$

pf $\text{Var}(X) = E(X^2) - E(X)^2 \rightarrow \left(\frac{1}{p}\right)^2$

$E(X^2) = \sum_{i=1}^{\infty} i^2 P(X=i) \rightarrow$ "law of the unconscious statistician"
(sum over values i of X and use $P(X=i)$)

$= \sum_{i=1}^{\infty} i^2 (1-p)^{i-1} p \rightarrow$ requires another trick

$$\begin{aligned}
E(X^2) &= 1\cdot p + 4\cdot(1-p)p + 9\cdot(1-p)^2\cdot p + \ldots \\
-(1-p)E(X^2) &= -0 - 1\cdot(1-p)\cdot p - 4\cdot(1-p)^2\cdot p - \ldots \\
\hline
&= \underbrace{1}_{2\cdot1-1}\cdot p + \underbrace{3}_{2\cdot2-1}\cdot(1-p)\cdot p + \underbrace{5}_{2\cdot3-1}\cdot(1-p)^2\cdot p + \ldots
\end{aligned}$$

$$= \sum_{i=1}^{\infty} \underbrace{\left(i^2 - (i-1)^2\right)}_{= i^2 - (i^2 - 2i + 1) = 2i-1} \cdot (1-p)^{i-1} \cdot p = \sum_{i=1}^{\infty} (2i-1)(1-p)^{i-1} p$$

$\Rightarrow E(X^2) - (1-p)E(X^2) = \sum_{i=1}^{\infty} (2i-1)(1-p)^{i-1} p$

<u>Thm</u> If $X \sim \text{Geometric}(p)$ then $\text{Var}(X) = \frac{1-p}{p^2}$

<u>pf</u> (continued) $\text{Var}(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2}$

$$= \frac{2-p-1}{p^2} = \frac{1-p}{p^2}$$

Have: $E(X^2) - (1-p)E(X^2) = \sum_{i=1}^{\infty} (2i-1)(1-p)^{i-1} p$

$$= \underbrace{2\sum_{i=1}^{\infty} i(1-p)^{i-1} p}_{E(X)} - \underbrace{\sum_{i=1}^{\infty} (1-p)^{i-1} p}_{1}$$

$$= 2\left(\frac{1}{p}\right) - 1 = \frac{2-p}{p}$$

$\Rightarrow E(X^2) - (1-p)E(X^2) = \frac{2-p}{p}$

$\Rightarrow E(X^2)(1-(1-p)) = \frac{2-p}{p}$

$\Rightarrow E(X^2) = \frac{2-p}{p} \cdot \frac{1}{1-(1-p)} = \frac{2-p}{p^2}$

<u>Comment</u> Sometimes to prove things, you need to get your hands dirty

② Poisson distribution

Poisson ≠ Poison  vs. 

λ can be any positive real number

Def A random variable X has the Poisson distribution with parameter λ if the distribution of X is

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k = 0, 1, 2, \ldots$$

we write $X \sim \text{Poisson}(\lambda)$ to indicate this

Comment

Each one has a natural model. What about Poisson?

- Bernoulli — Flip a biased coin. 1 if heads, 0 if tails
- Binomial — Flip a biased coin n times & count # of heads
- Geometric — Flip a biased coin until you get heads. Count # of flips
- Poisson — ??

2.1 Intuitive meaning of Poisson distribution

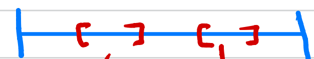
Poisson distribution: $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$

Intuitive meaning

Some event occurs at a constant rate of λ times per minute

or other unit of time

# of times it occurs in any 2 disjoint intervals are independent



independent & same rate

X = # of times event occurs in 1 minute

What random variables have Poisson distribution?

empirical question

1. Number of mutations in the DNA of a cell in 1 year
2. Number of requests to a server per day

Are these really Poisson distributions? Maybe not, but it can be a good model of both

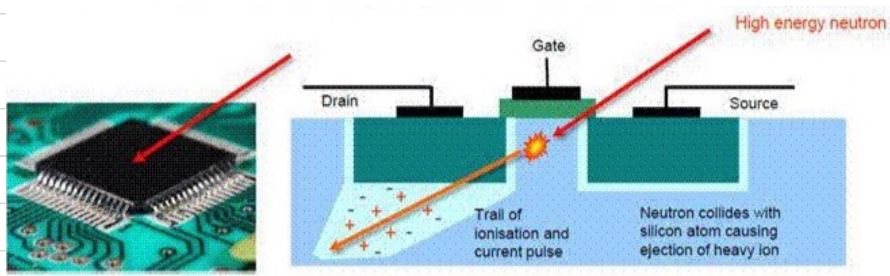
3. Number of raindrops in a day

Not Poisson distribution. Events not independent.

A cool example

Sometimes, cosmic rays pass through your computer

When they do, they can cause a bit stored in RAM to flip from 0 to 1 or 1 to 0



This is a real problem!

How many bits do we expect to flip this way per year?

Can model this using a Poisson distribution

Poisson distribution: $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ for $k = 0, 1, 2, \ldots$

<u>Uses of the Poisson distribution</u>

Use #1: Model various naturally occurring situations in science, technology, etc Especially rare events

Use #2: Approximation to the binomial distribution

Suppose $X \sim \text{Bin}(n, p)$

np^2 small $\Rightarrow$ distribution of $X \approx \text{Poisson}(\underbrace{np}_{\text{expected value of } X})$

One formalization of this:

<u>Thm</u> For $n = 1, 2, 3, \ldots$ let $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$

Then for any $k \in \mathbb{N}$

$$\lim_{n \to \infty} P(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$\leftarrow$ will prove in a few slides

2.2 Calculating things about the Poisson distribution

Suppose $X \sim \text{Poisson}(\lambda)$

What are $E(X)$ and $Var(X)$

Warm-up $\sum_{k=0}^{\infty} P(X=k) = 1$

pf
$$\sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda}$$
$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$
$$= e^{-\lambda} \cdot e^{\lambda} = e^{-\lambda+\lambda} = e^0 = 1$$

Comment Taylor series for e^x:
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Lesson: When working with Poisson distribution, useful to know things about e^x

Thm If $X \sim \text{Poisson}(\lambda)$ then $E(X) = \lambda$

↳ expected # of events per unit time = rate of occurrences

pf $E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k)$

$= \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} = e^{-\lambda}(\lambda e^{\lambda})$

$= \lambda e^{-\lambda+\lambda}$

$= \lambda e^0$

$= \lambda$

Consider $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$\frac{d}{dx} e^x = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$e^x = \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k!}\right) = \sum_{k=0}^{\infty} k \cdot \frac{x^{k-1}}{k!}$ (when $x \neq 0$)

$\Rightarrow xe^x = x \sum_{k=0}^{\infty} k \cdot \frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} k \cdot \frac{x^k}{k!}$

Plug in $x = \lambda$ $\lambda e^{\lambda} = \sum_{k=0}^{\infty} k \cdot \frac{x^k}{k!}$

Thm If $X \sim \text{Poisson}(\lambda)$ then $Var(X) = \lambda$ → same as $E(X)$

pf $Var(X) = E(X^2) - \underbrace{E(X)^2}_{=\lambda^2} = (\lambda^2+\lambda) - \lambda^2 = \lambda$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P(X=k) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!}$$

$$= e^{-\lambda}(\lambda^2 e^{\lambda} + \lambda e^{\lambda})$$

$$= \lambda^2 + \lambda$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \frac{d^2}{dx^2} e^x = \frac{d^2}{dx^2} \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= \sum_{k=0}^{\infty} k \cdot (k-1) \frac{x^{k-2}}{k!} \quad (\text{when } x \neq 0)$$

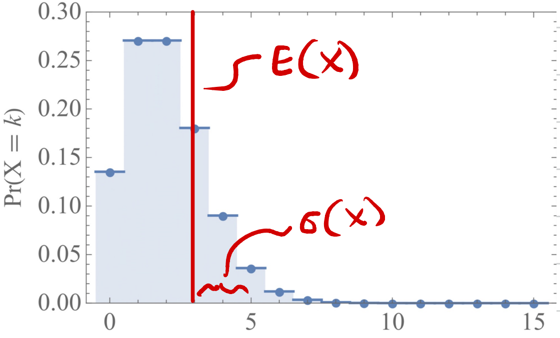
$$= \sum_{k=0}^{\infty} k^2 \frac{x^{k-2}}{k!} - \sum_{k=0}^{\infty} k \cdot \frac{x^{k-2}}{k!}$$

$$\Rightarrow x^2 e^x = \sum_{k=0}^{\infty} k^2 \frac{x^k}{k!} - \underbrace{\sum_{k=0}^{\infty} k \cdot \frac{x}{k!}}_{= xe^x} = \sum_{k=0}^{\infty} k^2 \frac{x^k}{k!} - xe^x$$

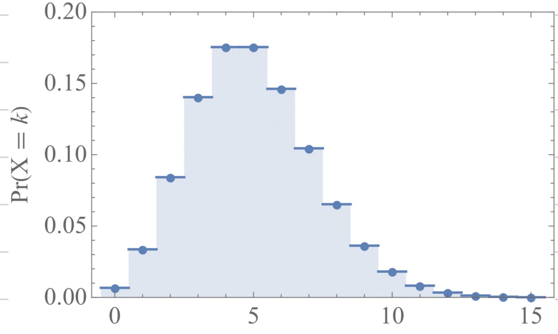
Plug in $x = \lambda$ $\quad \lambda^2 e^{\lambda} = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} - \lambda e^{\lambda}$

Comment Common way to check if Poisson distribution is a good model for a situation: Check mean = variance

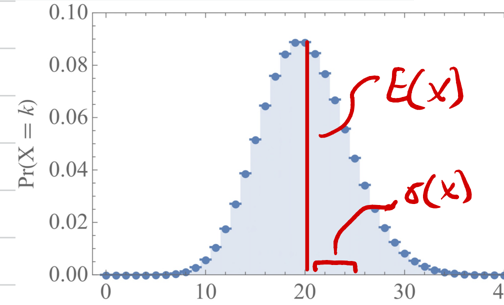
What does Poisson(λ) look like?



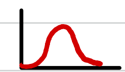
$\lambda = 3$



$\lambda = 5$



$\lambda = 20$

As $\lambda \to \infty$, approaches normal distribution 

Later we will see this is explained by the following fact

<u>Thm</u> If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent
then $X + Y \sim \text{Poisson}(\lambda + \mu)$

Thm If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ are independent then $X+Y \sim \text{Poisson}(\lambda+\mu)$

pf Let $k \in \mathbb{N}$.
Need to show $P(X+Y=k) = \frac{(\lambda+\mu)^k}{k!} e^{-(\lambda+\mu)}$

$$P(X+Y=k) = \sum_{i=0}^{k} P(X=i \text{ and } Y=k-i) \quad \text{law of total probability}$$

$$= \sum_{i=0}^{k} P(X=i)\, P(Y=k-i) \quad \text{independence}$$

$$= \sum_{i=0}^{k} \frac{\lambda^i}{i!} e^{-\lambda} \frac{\mu^{k-i}}{(k-i)!} e^{-\mu}$$

$$= e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{1}{i!\,(k-i)!} \lambda^i \mu^{k-i}$$

$$= k!\, e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \lambda^i \mu^{k-i}$$

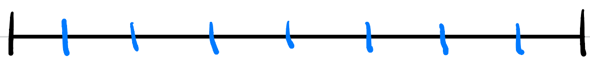
$$= k!\, e^{-(\lambda+\mu)} (\lambda+\mu)^k \quad \checkmark$$

Thm For $n = 1, 2, 3, \ldots$ let $X_n \sim \mathrm{Bin}(n, \frac{\lambda}{n})$

Then for any $k \in \mathbb{N}$

$$\lim_{n \to \infty} P(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Intuition Poisson(λ)



expected number of events: λ

① Divide into n parts

② Approximate by flipping a biased coin to decide if the event happens in each part

③ Expected number of events in each part: $\frac{\lambda}{n}$

↳ $\mathrm{Bin}(n, \frac{\lambda}{n})$. As $n \to \infty$, approximation gets better

Thm For $n=1,2,3,\ldots$ let $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$

Then for any $k \in \mathbb{N}$

$$\lim_{n\to\infty} P(X_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Pf

$$P(X_n = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n\cdot(n-1)\cdots(n-k)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!}\underbrace{\left(\frac{n\cdot(n-1)\cdots(n-k)}{n^k}\right)}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}} = \frac{\lambda^k}{k!}e^{-\lambda}$$

$$\lim_{n\to\infty} \frac{n}{n}\cdot\frac{n-1}{n}\cdot\cdots\cdot\frac{n-k}{n} = 1\cdot 1\cdots 1 = 1$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k} = (1-0)^{-k} = 1$$

③ Summary

Bernoulli $X \sim Bernoulli(p)$

$E(X) = p$
$Var(X) = p(1-p)$

Binomial $X \sim Bin(n, p)$

$E(X) = np$
$Var(X) = np(1-p)$

Geometric $X \sim Geometric(p)$

$E(X) = \frac{1}{p}$
$Var(X) = \frac{1-p}{p^2}$

Poisson $X \sim Poisson(\lambda)$

$E(X) = \lambda$
$Var(X) = \lambda$