1. Intro to Continuous Probability
2. Probability Density Function (PDF)
3. Continuous Uniform R.V. (running example)
4. Analogs Between Continuous & Discrete
5. Cumulative Distribution Function (CDF)
6. Conditioning on an Event.
7. Mixed Random Variables
8. Conditional Expectation
9. Two Envelope Paradox.

<u>Announcements</u>

- Midsemester Survey Feedback.

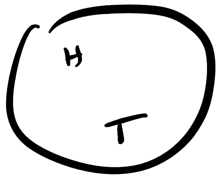
  Homework is hard and takes lots of time but really helps you learn!

Continous Probability

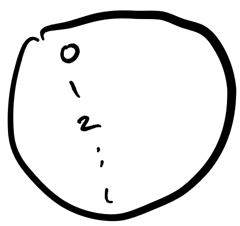
Random Variable X:

function $X : \Omega \to \mathbb{R}$

$X = k$ corresponds to $\{\omega \in \Omega : X(\omega) = k\}$

Coin flip $\rightarrow$ 

$|\Omega| = 2$

Poisson RV $\rightarrow$ 

$|\Omega| = |\mathbb{N}|$

In the real world, we are often more intrested in situations where Ω is uncountably infinite in size

Example Situation

Probability Density Function

A probability density function (pdf) for a real-valued r.v. X is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

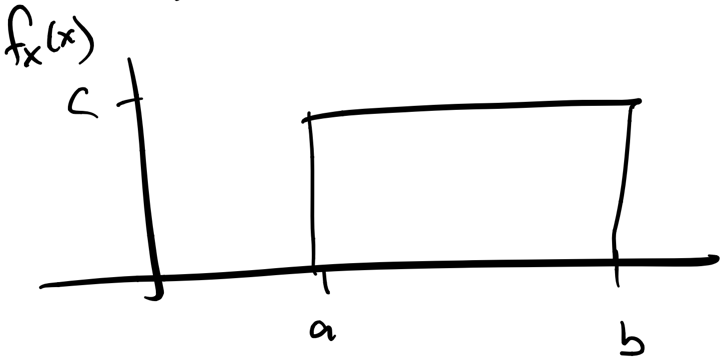
and:

Note:

Probability Density?

Example: $X \sim \text{Unif}(0,2)$

Example: $X \sim \text{Unif}(a, b)$



$$f(x) = \begin{cases} 0 & \forall\, x \in (-\infty, a) \\ \dfrac{1}{b-a} & \forall\, x \in [a, b] \\ 0 & \forall\, x \in (b, \infty) \end{cases}$$

What does uniformly random mean?

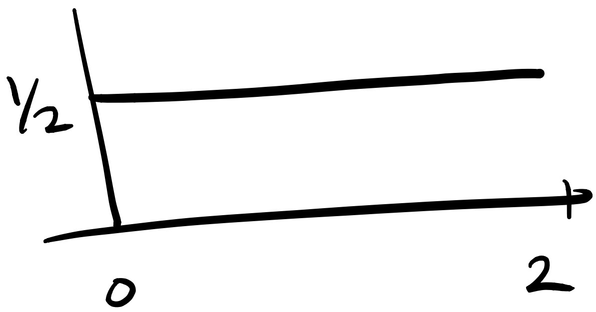
Analogs

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left(\int_{-\infty}^{\infty} x f(x)\, dx\right)^2$$

Example $X \sim \text{Unif}(0,2)$



What is $\mathbb{E}[X]$?

Cumulative Distribution Function (CDF)

The CDF is a bridge between discrete and continuous cases.

The cumulative distribution function (CDF) of a random variable X is the function F where:

Example: $X \sim \text{Unif}(0,2)$

What is the CDF of X?

Cumulative Distribution Function

The CDF has the following key properties:

Example: $X \sim \text{Unif}(0,2)$

Recall: CDF of X

$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0 & \forall x < 0 \\ \frac{x}{2} & \forall x \in [0,2] \\ 1 & \forall x > 2 \end{cases}$$

Recovering PDF/PMF from CDF

Continuous Case:

$$f(x) = \frac{dF(x)}{dx} \qquad \text{(by FTC)}$$

Discrete Case:

$$\mathbb{P}(x) = \frac{F(x) - F(x-1)}{x - (x-1)}$$

Example: $X \sim \text{Unif}(0,2)$

CDF to PDF

$$\mathbb{P}(X \leq x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & x \in [0,2] \\ 1 & x > 2 \end{cases}$$

Discrete vs Contivous Recap:

Discrete	Continuous
X	X

Note:

Conditioning on an Event

X continuous R.V.

A event w/ $\mathbb{P}(A) > 0$

$\mathcal{A}$ set of values $X(\omega)$ for all $\omega \in A$

$$\mathbb{P}(X \in \mathcal{A}) = \mathbb{P}(A)$$

Example

Mixed Random Variables

Some random variables are neither discrete or continous, but rather a combination of the two.

Example:

You flip a fair coin. If it is heads, you get a reward of 0.5 points. If it is tails, you spin a wheel to get a point value in $[0,2]$

Let X be the mixed random variable representing the amount of points you have at the end of the game.

Conditional Expectation

Let A be an event, and X a continous random variable.

$$\mathbb{E}[X \mid A] =$$

This also holds in the discrete case, just use pmf instead of pdf.

It follows;

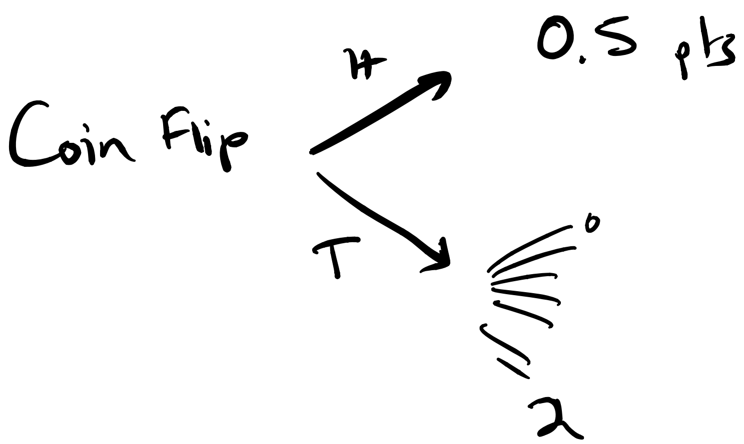
$$\mathbb{E}[X] = \mathbb{E}[X \mid A] \cdot \mathbb{P}(A) + \mathbb{E}[X \mid A^c] \cdot \mathbb{P}(A^c)$$

Proof left as exercise.

Hint: ① $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$

② Use law of total probability on $\mathbb{P}(X = x)$

Conditional Expectation Example.



What is $\mathbb{E}[X]$?

$\mathbb{E}[X \mid T] =$

$\mathbb{E}[X \mid H] =$

$\mathbb{E}[X] = \mathbb{E}[X \mid T]\,\mathbb{P}(T) + \mathbb{E}[X \mid H]\,\mathbb{P}(H)$

$=$

$=$

$=$

Two Envelope Paradox:

- Identical in appearance, weight, etc.
- One envelope has \$ x the other has \$ $2x$.
  - x is a positive real number

You are given an unopened envelope at random.

You then get a chance to keep it (and the money inside) or switch to the other one.

What should you do?

Argument 1: It doesn't matter (by symmetry)

Argument 2:

Let A be the amount in the envelope you are given. Let B be the amount in the other one.

$$\mathbb{E}[B] =$$