1. Intro to Continuous Probability

2. Probability Density Function (PDF)

3. Continuous Uniform R.V. (running example)

4. Analogs Between Continuous & Discrete

5. Cumulative Distribution Function (CDF)

6. Conditioning on an Event.

7. Mixed Random Variables

8. Conditional Expectation

9. Two Envelope Paradox.

Announcements

- Midsemester Survey Feedback.

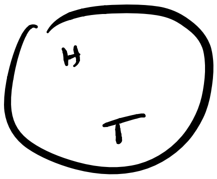
  Homework is hard and takes lots of time
  but really helps you learn!

Continous Probability

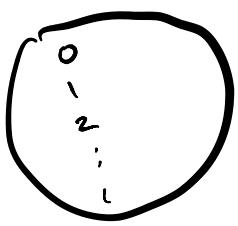
Random Variable X:

function $X : \Omega \to \mathbb{R}$

$X = k$ corresponds to $\{\omega \in \Omega : X(\omega) = k\}$

Coin flip $\rightarrow$ 

$|\Omega| = 2$

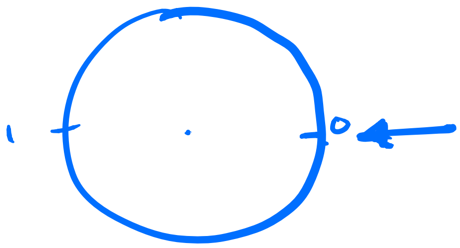
Poisson RV $\rightarrow$ 

$|\Omega| = |\mathbb{N}|$

In the real world, we are often more intrested in situations where Ω is uncountably infinite in size

Example Situation

Circumference: 2

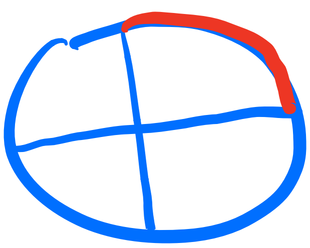


X is a continous R.V.

$X \in [0, 2)$

$\underline{\mathbb{P}(X = 0)} = 0$

If we assign $\mathbb{P}(X=0)$ a positive probability, then summing over all $k \in [0,2)$ gives us something bigger than 1



$\mathbb{P}(X \text{ lands on } \ \ ) = 1/4$

1) For a continuous random variable,

$$\mathbb{P}(X = \omega) = 0 \quad \text{for all } \omega \in \Omega$$

2) Intervals do have probability.

Probability Density Function

A probability density function (pdf) for a real-valued r.v. X is a function $f: \mathbb{R} \to \mathbb{R}$ satisfying:

1. $\forall x \in \mathbb{R}, \quad f(x) \geq 0$ (nonnegative)

2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

and:

$$\boxed{3.\quad \mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx}$$

Note:

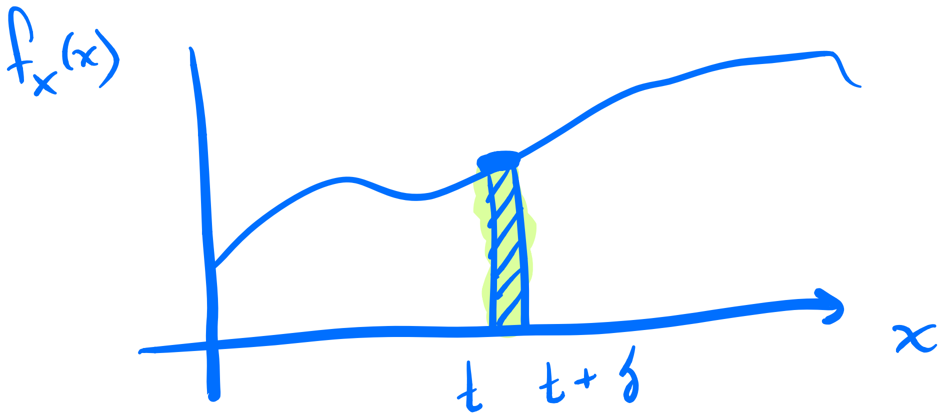
$$\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f(x)\,dx = 0$$

This implies that

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < x < b)$$

(endpoints don't contribute probability)

Probability Density?



Observe: If $\delta > 0$ is very small, then

$f_X(x)$ is roughly constant on $[t, t+\delta]$

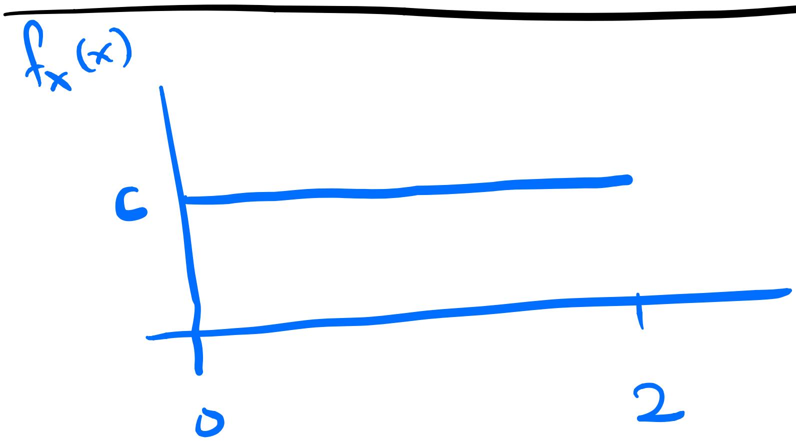
$$\mathbb{P}(t \le X \le t+\delta) = \int_t^{t+\delta} f_X(x)\,dx$$

$$\boxed{\approx f_X(t)\cdot\delta}$$

$$f_X(t) = \frac{\mathbb{P}(t \le X \le t+\delta)}{\delta} \qquad \begin{matrix} \leftarrow \text{probability} \\ \leftarrow \text{length} \end{matrix}$$

"probability density"

Example: $X \sim \text{Unif}(0,2)$



$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ c & \text{if } x \in [0,2] \\ 0 & \text{if } x > 2 \end{cases}$$

What is the pdf of X?

We know by def. of pdf.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

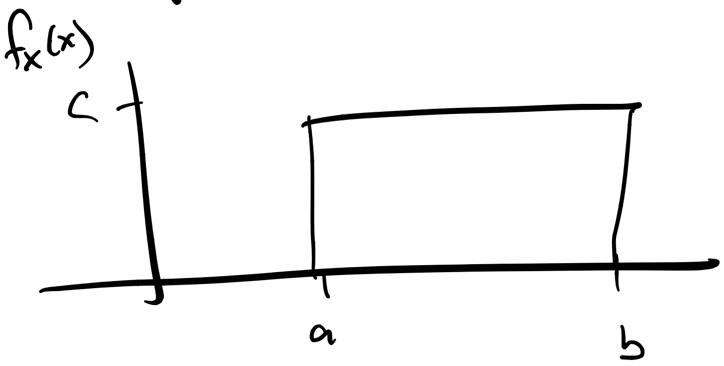
$$\int_0^2 c\,dx = 1$$

$$2 \cdot c = 1$$

$$c = 1/2$$

$$\boxed{f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x \in [0,2] \\ 0 & \text{if } x > 2 \end{cases}}$$

Example: $X \sim \text{Unif}(a, b)$



$$f_X(x) = \begin{cases} 0 & \forall\, x \in (-\infty, a) \\ \frac{1}{b-a} & \forall\, x \in [a, b] \\ 0 & \forall\, x \in (b, \infty) \end{cases}$$

What does uniformly random mean?

It means that probability is proportional to length.

Analogs

Summations $\rightarrow$ Integrals

PMF $\rightarrow$ PDF

LOTUS

$\mathbb{E}[g(x)]$

$= \sum_{x} g(x) \cdot \mathbb{P}(X=x)$

$$\mathbb{E}[x] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$Var(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

LOTUS (continous)

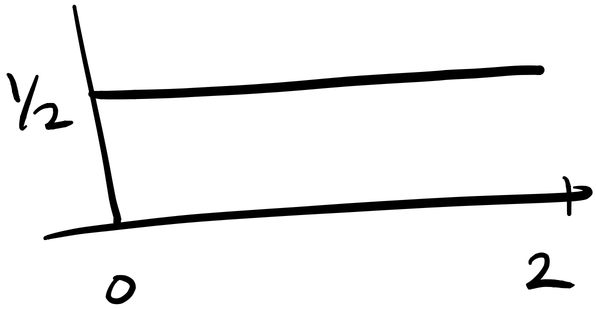
$$= \int_{-\infty}^{\infty} x^2 \underbrace{f(x)}_{\text{pdf of } x}\, dx - \left(\int_{-\infty}^{\infty} x f(x)\, dx\right)^2$$

PMF for poisson r.v:

$$\mathbb{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Probability mass function

Example $X \sim \text{Unif}(0, 2)$



What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_0^2 x \cdot \tfrac{1}{2}\, dx = \frac{x^2}{2} \cdot \frac{1}{2} \Big|_0^2 = \frac{4}{4} - 0 = 1$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx = \int_0^2 x^2 \cdot \tfrac{1}{2}\, dx = \frac{8}{6}$$

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

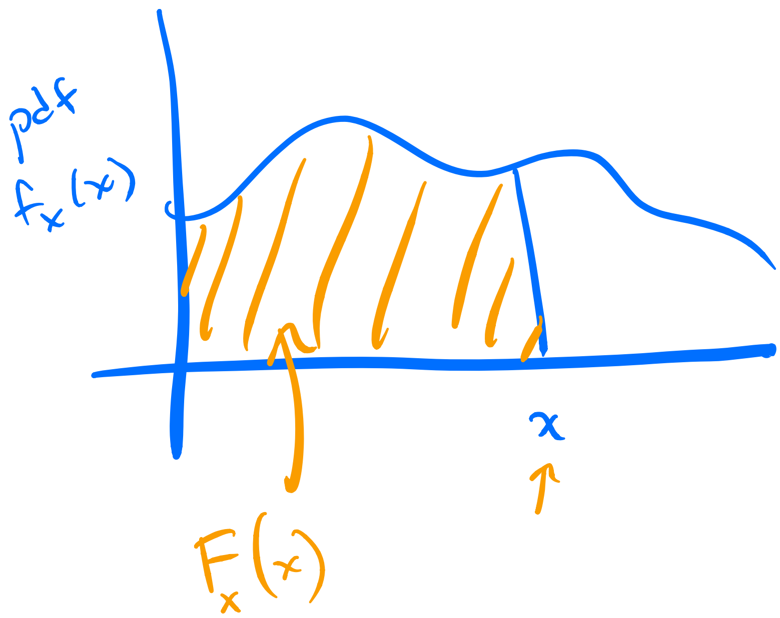
$$= \frac{8}{6} - 1^2 = \frac{1}{3}$$

Cumulative Distribution Function (CDF)

The CDF is a bridge between discrete and continuous cases.

The cumulative distribution function (CDF) of a random variable X is the function F where:

$$F(x) = \mathbb{P}(X \leq x)$$



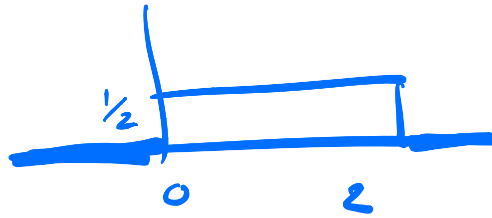
CDF is always a probability.

In the continuous case,

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

Example: $X \sim \text{Unif}(0,2)$

What is the CDF of X?



$$F(x) = \mathbb{P}(X \le x)$$

$$= \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_0^x \frac{1}{2}\,dx = \frac{1}{2}x \Big|_0^x = \frac{x}{2}$$

(if $x \in [0,2]$)

$$\mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{2} & \text{if } x \in [0,2] \\ 1 & \text{if } x > 2 \end{cases}$$

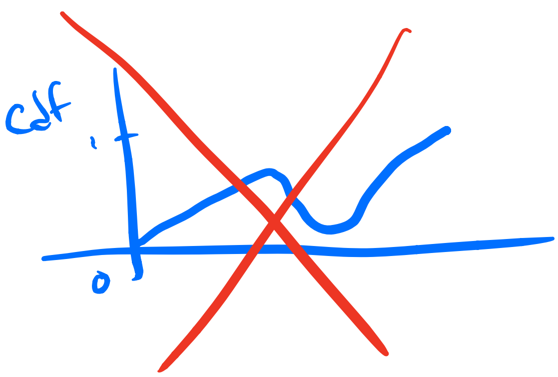
Cumulative Distribution Function

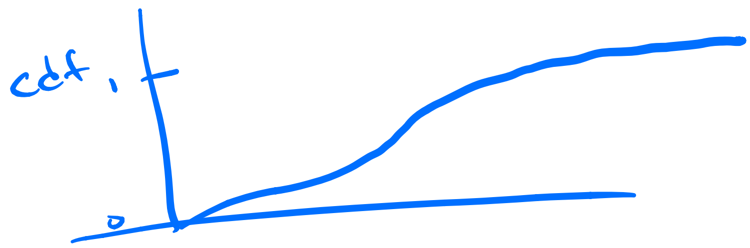
The CDF has the following key properties:

1. $\lim_{x \to -\infty} F(x) = 0$

2. $\lim_{x \to +\infty} F(x) = 1$

3. $F(x)$ is monotonically increasing





4. $F(x)$ uniquely characterizes the distribution of X

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

(by FTC)

Example: $X \sim \text{Unif}(0,2)$

Recall: CDF of X

$$F(x) = \mathbb{P}(X \le x) = \begin{cases} 0 & \forall x < 0 \\ \frac{x}{2} & \forall x \in [0,2] \\ 1 & \forall x > 2 \end{cases}$$

$$\lim_{x \to -\infty} F(x) = 0 \quad \checkmark$$

$$\lim_{x \to +\infty} F(x) = 1 \quad \checkmark$$

Monotonically increasing $\checkmark$

$$\boxed{\text{Break: 4:06 PM}}$$

Recovering PDF/PMF from CDF

Take a derivative.

Continuous Case:

$$f(x) = \frac{dF(x)}{dx} \quad (\text{by } \underline{\underline{\text{FTC}}})$$

Discrete Case:

$$\mathbb{P}(x) = \frac{F(x) - F(x-1)}{x - (x-1)}$$

Example: $X \sim \text{Unif}(0,2)$

CDF to PDF

$$\mathbb{P}(X \leq x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & x \in [0,2] \\ 1 & x > 2 \end{cases}$$

$\downarrow$ derivative $\quad$ $\uparrow$ integrate.

$$f(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x \in [0,2] \\ 0 & x > 2 \end{cases}$$

Discrete vs Contiuous Recap:

Discrete	Continuous
X	X
PMF $\mathbb{P}(X=x)$ ("mass")	PDF $f_X(x)$ ("density")
CDF $\mathbb{P}(X \leq x)$	CDF $\mathbb{P}(X \leq x)$
$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X=x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$
$\vdots$	$\vdots$

Note:

The PDF of continuous random variable can be greater than 1

Consider $X \sim \text{Unif}[0, \frac{1}{2}]$

Conditioning on an Event

X continuous R.V.

$\mathcal{A}$ set of values $X(\omega)$

A event $\mathbb{P}(A) > 0$

A is the set of outcomes such that $X(w)$ is in fancy A

$$\mathbb{P}(X \in \mathcal{A}) = \mathbb{P}(A)$$

Note: This has been corrected. By defining the fancy A first as a set of values the random variable could take on, and then A as all the outcomes that result in a value in the set fancy A, there are no outcomes outside of A that result in a value in fancy A, so the two probabilities are equal.

$$f_{X|A}(x) \cdot \cancel{\delta} = \mathbb{P}(x \le X \le x+\delta \mid X \in \mathcal{A})$$

$$= \frac{\mathbb{P}(x \le X \le x+\delta \cap X \in \mathcal{A})}{\mathbb{P}(X \in \mathcal{A})}$$

$$\approx \begin{cases} \dfrac{f_X(x) \cdot \cancel{\delta}}{\mathbb{P}(A)} & \text{if } x \in \mathcal{A} \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow f_{X|A} = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(A)} & \forall x \in \mathcal{A} \\ 0 & \text{o.w.} \end{cases}$$

Discrete Case:

If X is a discrete R.V. $\mathbb{P}_{X|A}(x) = \begin{cases} \dfrac{\mathbb{P}(X=x)}{\mathbb{P}(A)} & \text{if } x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$

Example

$f_X(x)$ is the pdf of X

$\boxed{X \sim \text{Unif}[0,2]}$

Let A be the event $X \in [0,1]$

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(A)} & \text{if } \underline{x \in [0,1]} \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \dfrac{1/2}{1/2} & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

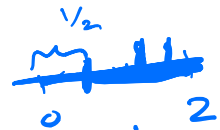
Mixed Random Variables

Some random variables are neither discrete or continuous, but rather a combination of the two.

$1/2$

Example:

You flip a fair coin. If it is heads, you get a reward of 0.5 points. If it is tails, you spin a wheel to get a point value in $[0, 2]$



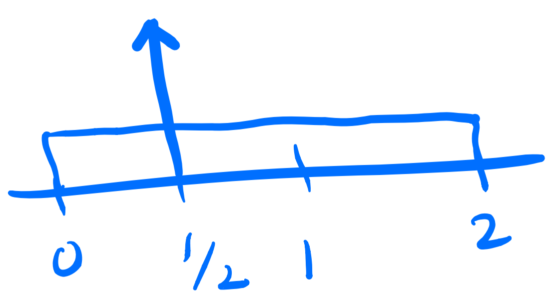
$\frac{1/2}{2}$

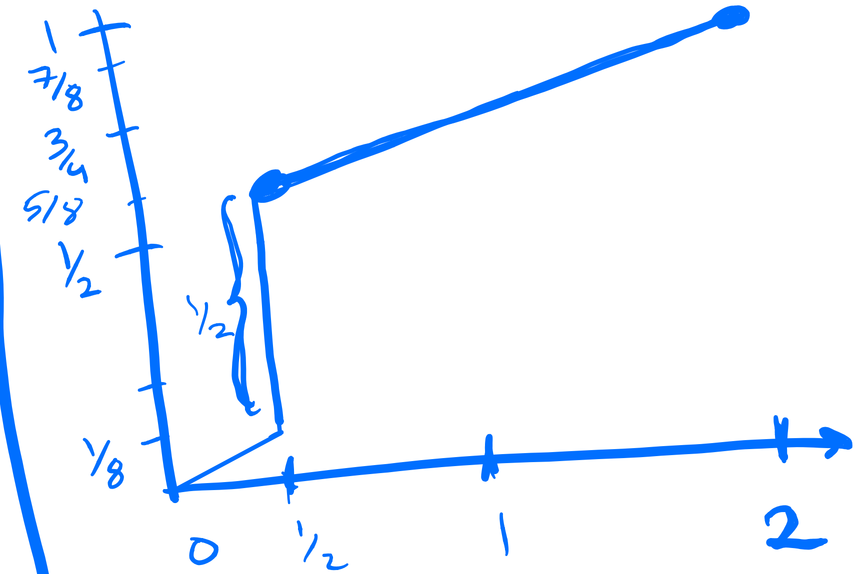
Let X be the mixed random variable representing the amount of points you have at the end of the game.

CDF

$F(x) = P(X \le x)$

PMF / PDF





$\boxed{P(X < 1/2)}$ → Getting tails, fall between $[0, 1/2]$

$X \le 1/2$ → Also can now get heads

Conditional Expectation

Let A be an event, and X a continous random variable.

$$\mathbb{E}[X \mid A] =$$

This also holds in the discrete case, just use pmf instead of pdf.

It follows;

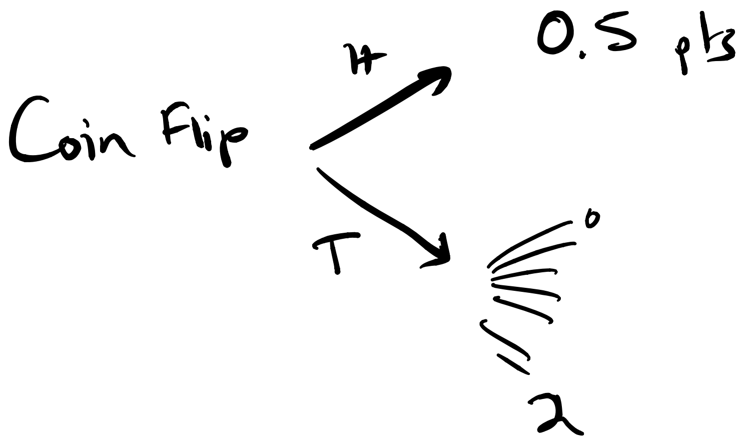
$$\mathbb{E}[X] = \mathbb{E}[X \mid A] \cdot \mathbb{P}(A) + \mathbb{E}[X \mid A^c] \cdot \mathbb{P}(A^c)$$

Proof left as exercise.

Hint: (1) $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x)$

(2) Use law of total probability on $\mathbb{P}(X = x)$

Conditional Expectation Example.



What is $\mathbb{E}[X]$?

$\mathbb{E}[X|T] =$

$\mathbb{E}[X|H] =$

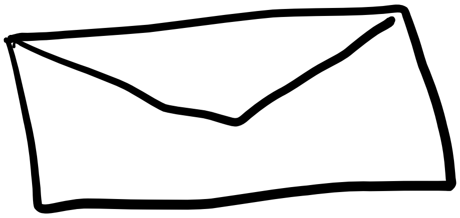
$\mathbb{E}[X] = \mathbb{E}[X|T]\,\mathbb{P}(T) + \mathbb{E}[X|H]\,\mathbb{P}(H)$

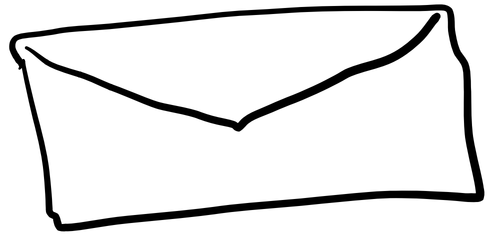
$=$

$=$

$=$

Two Envelope Paradox:





- Identical in appearance, weight, etc.
- One envelope has $\$x$ the other has $\$2x$.
  - x is a positive real number

You are given an unopened envelope at random.

You then get a chance to keep it (and the money inside) or switch to the other one.

What should you do?

Argument 1: It doesn't matter (by symmetry)

Argument 2:

Let A be the amount in the envelope you are given. Let B be the amount in the other one.

$\mathbb{E}[B] =$