1. Conditional Expectation
2. Two Envelope Paradox.
3. Exponential Distribution
4. Memoryless Property
5. Tail Sum Formula
6. Two Envelope Paradox Revisited.

Discrete vs Contivous Recap:

Discrete	Continuous
X	X
PMF $\mathbb{P}(X=x)$ (mass)	PDF $f_X(x)$ (density)
CDF $\mathbb{P}(X \leq x)$	CDF $\mathbb{P}(X \leq x)$
$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X=x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

Note:

The PDF of a continous R.V. can be greater than 1

Consider $X \sim \text{Unif}[0, \frac{1}{2}]$

Mixed Random Variables

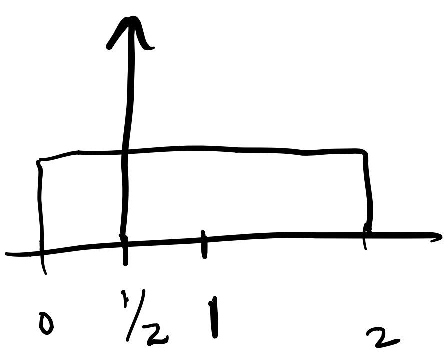
Some random variables are neither discrete or continuous, but rather a combination of the two.

Example:

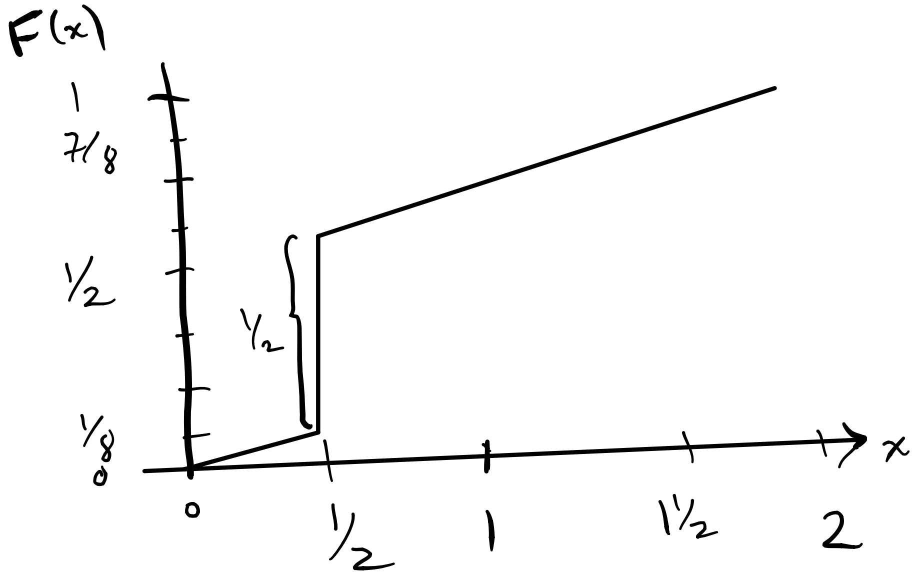
You flip a fair coin. If it is heads, you get a reward of 0.5 points. If it is tails, you spin a wheel to get a point value in [0, 2]

Let X be the mixed random variable representing the amount of points you have at the end of the game.

PMF / PDF



CDF



Conditional Expectation

Let A be an event, and X a continous random variable.

$$\mathbb{E}[X|A] =$$

This also holds in the discrete case, just use pmf instead of pdf.

It follows;

$$\mathbb{E}[X] =$$

Proof left as exercise.

Hint: (1) $\mathbb{E}[X] = \sum_{x} x \cdot \mathbb{P}(X=x)$

(2) Use law of total probability on $\mathbb{P}(X=x)$

Conditional Expectation Example.

Two Envelope Paradox:

- Identical in appearance, weight, etc.
- One envelope has $x the other has $2x.
  - x is a positive real number

You are given an unopened envelope at random, and can see how much money is inside.
You then get a chance to keep it (and the money inside) or switch to the other one.

What should you do?

Exponential Distribution

The exponential distribution is the continuous analog of the geometric distribution.
In the case of the geometric coin flipping experiment, we know the first Heads happens at a discrete trial / flip number.

In the real world, we might be waiting for a system to crash, or for a Piazza question to be answered. Here we are waiting for a point in continuous time. In such scenarios, the exponential distribution is a natural fit.

For $\lambda > 0$, a continuous random variable X with pdf

$$f(x) =$$

is called an exponential random variable with rate parameter λ, and we write $X \sim Exp(\lambda)$

Note: ① Geometric distribution also has a single parameter p

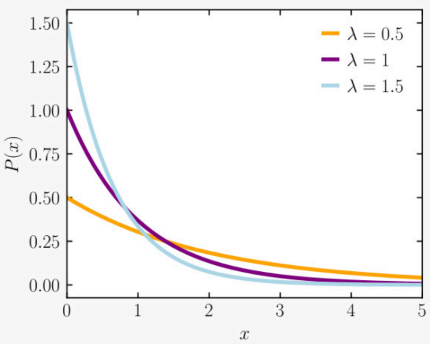
② λ is the "success rate"

Example

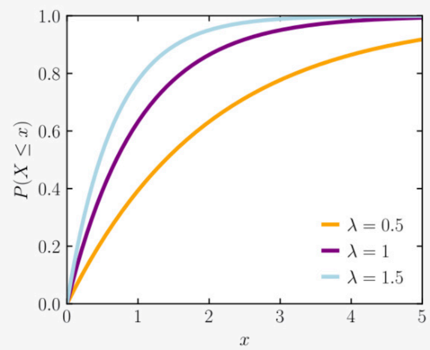
You are getting phone calls at a rate of about 2 calls per hour.
Then, you may wish to model the amount of time until the next call as Exp(2)

Exponential

Probability density function



Cumulative distribution function



Parameters	$\lambda > 0$, rate, or inverse scale
Support	$x \in [0, \infty)$
PDF	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$

Check

Mean & Variance of an Exponential

Let $X \sim Exp(\lambda)$

$\mathbb{E}[X] =$

$\mathbb{E}[X^2] =$

$Var[X] =$

$=$

Ex: 2 phone call per hour approx

$X \sim Exp(2)$

$\mathbb{E}[X] =$

CDF of an Exponential

$X \sim \text{Exp}(\lambda)$

If $x < 0$, $\mathbb{P}(X \leq x) = 0$

Otherwise,

$\mathbb{P}(X \leq x) =$

Continuous Analog of Geometric

How did they come up with the exponential r.v.?

Consider a discrete setting w/ 1 trial every δ seconds.

λ is our fixed rate of success per unit time : $\lambda = \frac{p}{\delta}$

So, success probability of a trial $p = \lambda \cdot \delta$

Let Y be the r.v. for the time until the first success

$P(Y > k \cdot \delta) =$

Memoryless Property (Also applies to Geometric R.V.)

What does memoryless mean?

"How long you have waited won't affect how much longer you have to wait."

Let $X \sim Exp(\lambda)$, then

$\mathbb{P}(X > x + t \mid X > t) =$

$=$

$=$

$=$

Tail Sum Formula

Let X be a random variable that only takes on values in $\mathbb{N}$. Then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

Proof:

Continuous Tail Sum Formula.

Let X be a nonnegative random Variable.

Then, $$E[X] = \int_0^\infty (1 - F_X(x))\, dx$$

Two Envelopes Revisited

Consider the following strategy:

Draw $t \sim Exp(2)$ Let m be the amount of money in your envelope If $m < t$, switch to the other envelope. Else, stick with your envelope

Cases: