① Conditional Expectation

② Two Envelope Paradox.

③ Exponential Distribution

④ Memoryless Property

⑤ Tail Sum Formula

⑥ Two Envelope Paradox Revisited.

Correction:

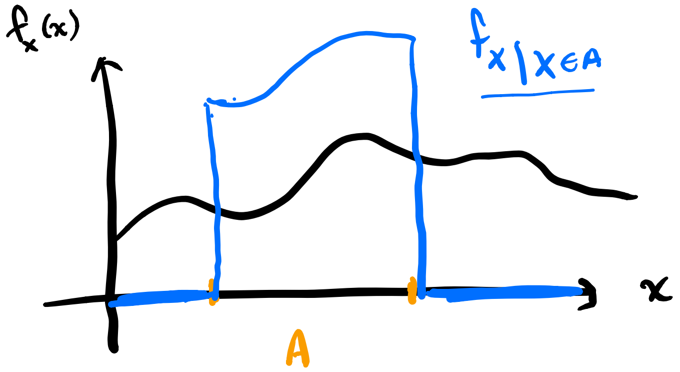
Let X be a continuous RV

Consider an event A with $P(A) > 0$.

$$\mathbb{P}(x \le X \le x+\delta \mid X \in A) \approx f_{X|X\in A}(x)\cdot\delta$$

$$= \frac{\mathbb{P}(x \le X \le x+\delta \cap X \in A)}{\mathbb{P}(A)} = \begin{cases} \frac{f_X(x)\cdot\delta}{\mathbb{P}(A)} & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|X\in A}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}(A)} & \text{if } X \in A \\ 0 & \text{o.w.} \end{cases}$$



Discrete vs Continuous Recap:

Discrete	Continuous
X	X
PMF $\mathbb{P}(X=x)$ (mass)	PDF $f_X(x)$ (density)
CDF $\mathbb{P}(X\le x)$	CDF $\mathbb{P}(X\le x)$
$\mathbb{E}[X]=\sum_x x\cdot\mathbb{P}(X=x)$	$\mathbb{E}[X]=\int_{-\infty}^{\infty} x\cdot f(x)\,dx$
$\vdots$	$\vdots$

Note:

The PDF of a continuous R.V. can be greater than 1

Consider $X \sim \text{Unif}[0, \frac{1}{2}]$

Mixed Random Variables

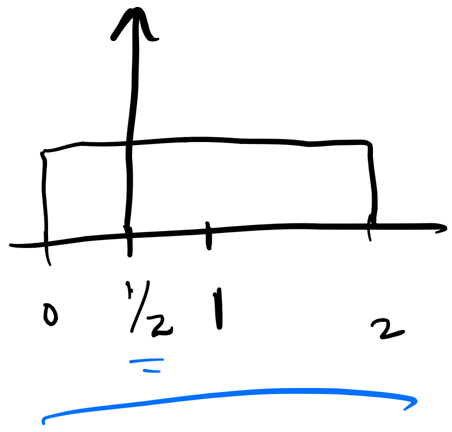
Some random variables are neither discrete or continous, but rather a combination of the two.

Example:

You flip a fair coin. If it is heads, you get a reward of 0.5 points. If it is tails, you spin a wheel to get a point value in [0, 2]

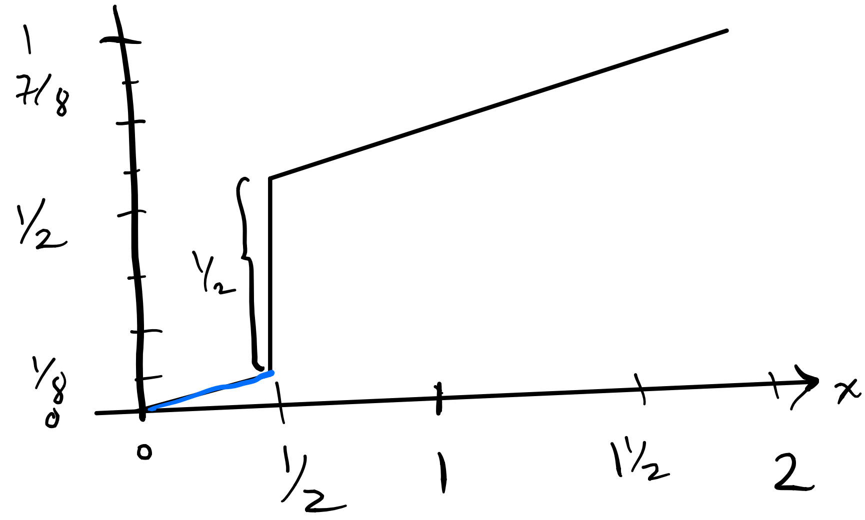
Let X be the mixed random variable representing the amount of points you have at the end of the game.

PMF / PDF



CDF

$F(x) = \mathbb{P}(X \le x)$



Conditional Expectation

Let A be an event, and X a continous random variable.

$$\mathbb{E}[X|A] = \int_{-\infty}^{\infty} x \cdot \underset{\substack{\uparrow \\ \text{conditional pdf}}}{f_{\underline{X|A}}}(x)\,dx$$

This also holds in the discrete case, just use <u>pmf</u> instead of pdf; and sum instead of integral

A^c is the complement of A

It follows:

$$\ast\ \mathbb{E}[X] = \mathbb{E}[X|A]\cdot\mathbb{P}(A) + \mathbb{E}[X|A^c]\cdot\mathbb{P}(A^c)$$

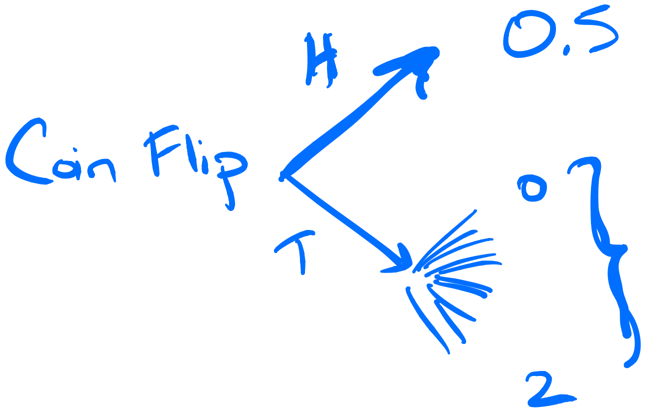
Total Probability, but for expectation.

Proof left as exercise.

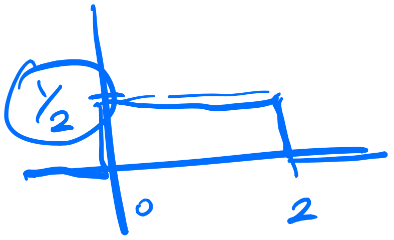
Hint: ① $\mathbb{E}[X] = \sum_{x} x \cdot \mathbb{P}(X=x)$

② Use law of total probability on $\mathbb{P}(X=x)$

Conditional Expectation Example.



What is $\mathbb{E}[X]$?



$$\mathbb{E}[X|H] = 0.5$$

Uniform$(0,2)$

$$\rightarrow \mathbb{E}[X|T] = \int_{-\infty}^{\infty} x \cdot p_{X|T}(x)\, dx$$

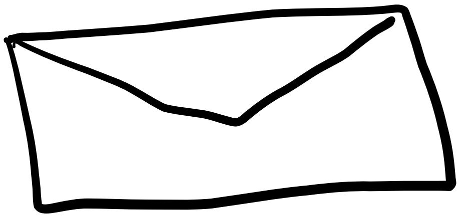
$$= \int_0^2 x \cdot \boxed{\tfrac{1}{2}}\, dx = 1$$

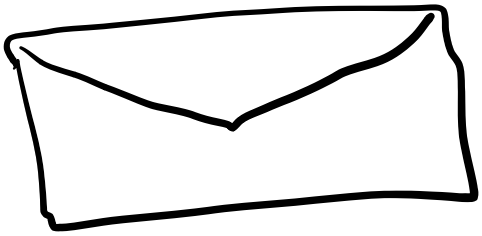
$$\mathbb{E}[X] = \mathbb{E}[X|T] \cdot \mathbb{P}(T) + \mathbb{E}[X|H] \cdot \mathbb{P}(H)$$

$$= 1 \cdot 0.5 + 0.5 \cdot 0.5$$

$$= 0.75$$

Two Envelope Paradox:





- Identical in appearance, weight, etc.
- One envelope has $\$x$ the other has $\$2x$.
  - x is a positive real number

You are given an unopened envelope at random, and can see how much money is inside.

You then get a chance to keep it (and the money inside) or switch to the other one.

What should you do?

Argument 1:

It doesn't matter if you switch, since by symmetry, you were just as likely to have been given the \$x envelope as compared to \$2x

Argument 2:

Let A be the amount in the envelope that you are given, and B be the amount in the other envelope.

$$\mathbb{E}[B] = \mathbb{E}[B \mid A<B]\cdot\mathbb{P}(A<B) + \mathbb{E}[B \mid A>B]\cdot\mathbb{P}(A>B)$$

$$= \mathbb{E}[B \mid B=2A]\cdot 1/2 + \mathbb{E}[B \mid B=\tfrac{A}{2}]\cdot 1/2$$

$$= \mathbb{E}[2A]\cdot 1/2 + \mathbb{E}[A/2]\cdot 1/2$$

$$= \mathbb{E}[A] + \frac{\mathbb{E}[A]}{4} = \frac{5}{4}\mathbb{E}[A]$$

So, we always switch?

Problem: The As are not the same.

$$= (2x)\cdot 1/2 + x\cdot(1/2) = 1.5x$$

$\mathbb{E}[A]$ is also $1.5x$

Exponential Distribution

The exponential distribution is the continuous analog of the geometric distribution. In the case of the geometric coin flipping experiment, we know the first Heads happens at a discrete trial / flip number.

In the real world, we might be waiting for a system to crash, or for a Piazza question to be answered. Here we are waiting for a point in continuous time. In such scenarios, the exponential distribution is a natural fit.

For $\lambda > 0$, a continuous random variable X with pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

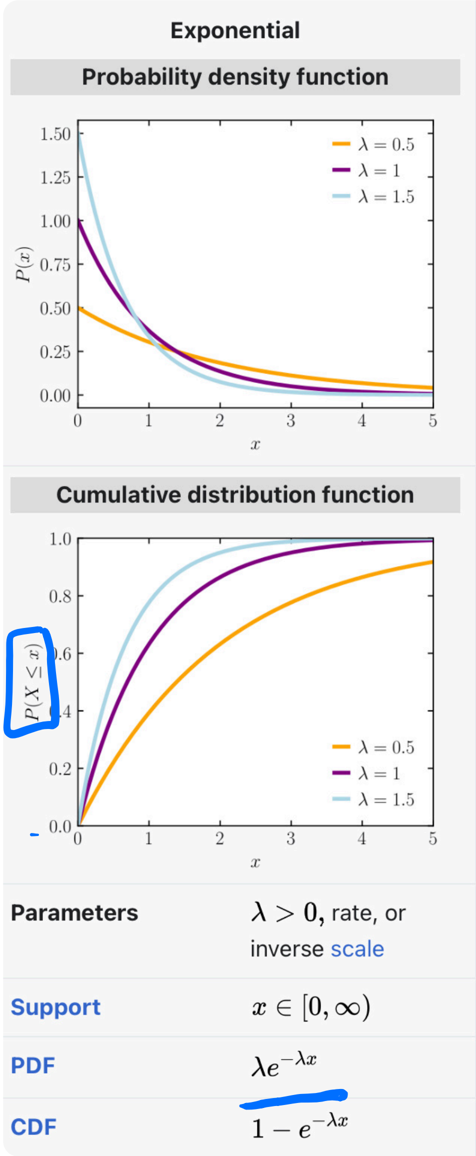
is called an exponential random variable with rate parameter λ, and we write $X \sim \text{Exp}(\lambda)$

Note: ① Geometric distribution also has a single parameter p

② λ is the "success rate"

Example

You are getting phone calls at a rate of about 2 calls per hour.
Then, you may wish to model the amount of time until the next call as Exp(2)



Exponential

Probability density function

Cumulative distribution function

Parameters	$\lambda > 0$, rate, or inverse scale
Support	$x \in [0, \infty)$
PDF	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$

From Wiki

Check

Is the exponential pdf valid?

① Is $f(x)$ nonnegative?

Yes.

② Does it integrate to 1?

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \lambda e^{-\lambda x} dx$$

$$= -e^{-\lambda x}\Big|_{0}^{\infty}$$

$$= 0-(-1) = 1$$

Yes.

Yes, the exponential pdf is valid.

Mean & Variance of an Exponential

Let $X \sim \text{Exp}(\lambda)$

$$\mathbb{E}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\,dx = \frac{2}{\lambda^2}$$

$$\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

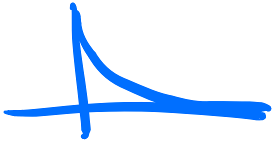
Ex: 2 phone call per hour approx

$X \sim \text{Exp}(2)$

$$\mathbb{E}[X] = \frac{1}{2}$$

Makes sense (expect a call every ½ hour)

CDF of an Exponential



$X \sim \text{Exp}(\lambda)$

If $x < 0$, $\mathbb{P}(X \le x) = 0$

Otherwise,

$$\mathbb{P}(X \le x) = \int_0^x \lambda e^{-\lambda s}\,ds = -e^{-\lambda s}\Big|_0^x = -e^{-\lambda x} - (-1)$$

$$= 1 - e^{-\lambda x}$$

The $\overbrace{\text{complement of the CDF}}^{\text{aka CCDF}}$ is $1 - \mathbb{P}(X \le x)$

$$\mathbb{P}(X > x) = 1 - \mathbb{P}(X \le x) = 1 - (1 - e^{-\lambda x}) = \underline{e^{-\lambda x}}$$

Note: The CCDF also uniquely identifies the distribution.

Continuous Analog of Geometric

How did they come up with the exponential r.v.?

Consider a discrete setting w/ 1 trial every δ seconds.

$\underline{\lambda}$ is our fixed rate of success per unit time : $\underline{\lambda} = \frac{p}{\delta}$

So, success probability of a trial $p = \lambda \cdot \delta$

Let Y be the r.v. for the time until the first success

$$P(Y > \underbrace{k \cdot \delta}_{}) = (1-p)^k = (1-\lambda\cdot\delta)^k$$

If we look in terms of time t,

$$P(Y > t) = P\left(Y > \underbrace{\left(\frac{t}{\delta}\right)}_{k} \cdot \delta\right) = \underline{(1-\lambda\cdot\delta)^{\frac{t}{\delta}}}$$

Take the limit as δ goes to 0,

$$P(Y > t) = \underbrace{e^{-\lambda t}}_{}$$

So, we have arrived at the exponential r.v.

Break : 4:05 PM.

Memoryless Property

(Also applies to Geometric R.V.)

What does memoryless mean?

"How long you have waited won't affect how much longer you have to wait."

Flipping coins until heads if you already flipped t tails.

Let $X \sim \text{Exp}(\lambda)$, then

$$\mathbb{P}(X > x + t \mid X > t) = \frac{P(X > x+t \wedge X > t)}{P(X > t)}$$

how much longer

how much we already waited

$$= \frac{P(X > x+t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda x} = \mathbb{P}(X \geq x)$$

No t involved!

Tail Sum Formula

Let X be a random variable that only takes on values in $\mathbb{N}$. Then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} P(X \geq k)$$

Proof:

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} \overbrace{x \cdot \mathbb{P}(X=x)}$$

mult as add.

$$= \boxed{\sum_{x=1}^{\infty} \sum_{k=1}^{x} \mathbb{P}(X=x)}$$

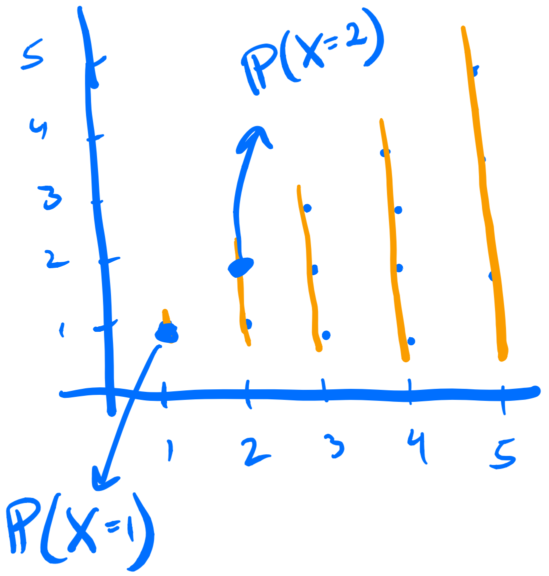
picture

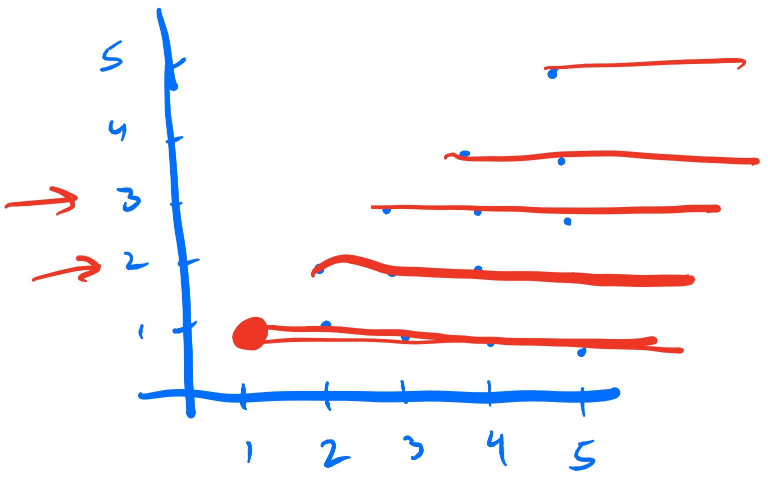
$$= \boxed{\sum_{k=1}^{\infty} \underbrace{\sum_{x=k}^{\infty} \mathbb{P}(X=x)}}$$

def.

$$= \sum_{k=1}^{\infty} \mathbb{P}(X \geq k)$$

- corresponds to $\mathbb{P}(X=x)$





Continuous Tail Sum Formula.

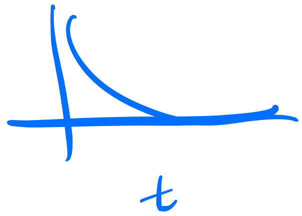
X is continuous.

Let X be a nonnegative random Variable.

Then, $E[X] = \int_0^\infty (1 - F_X(x))\, dx$

CDF of X

Two Envelopes Revisited



Consider the following strategy:

Draw $\underline{\underline{t}} \sim \underline{\text{Exp}(2)}$

Let m be the amount of money in your envelope

If $\underline{m < t}$, switch to the other envelope.

Else, stick with your envelope

Cases:

① $t < x$ and $t < 2x$ (m $\underline{\text{always bigger}}$ than t)

→ strategy does not help or hurt.

② $t > x$ and $t > 2x$ (m is $\underline{\text{always less}}$ than t!)

→ strategy does not help or hurt.

③ $x < t < 2x$

→ strategy only helps you.

If $m = x$, you switch

If $m = 2x$, you dont switch

} strategy is perfect.

$\mathbb{P}(x<t<2x)$ happens with positive probability.

Note:

The t "threshold" doesn't need to come from a particular distribution, so long as

$$P(x<t<2x) > 0$$

for any $x \in \mathbb{R}$ where $x>0$.

$$t \sim Exp(2)$$

$$\mathbb{P}(x<t<2x) = \int_x^{2x} 2e^{-2x}\,dx$$

$\uparrow$

pdf of $Exp(2)$

$$> 0$$