1. Normal (aka Gaussian) Distribution

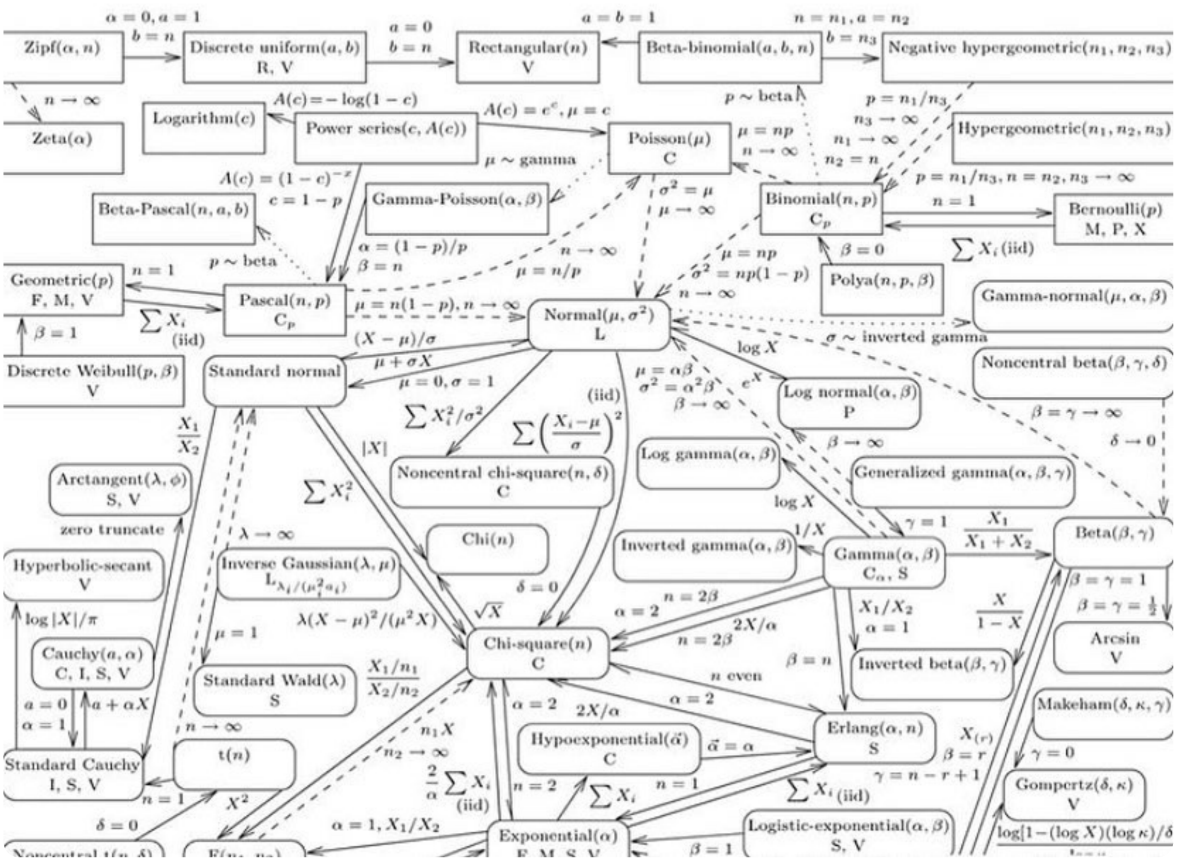
2. Standard Normal Distribution

3. Standard Normal CDF $\Phi()$

4. Central Limit Theorem

Teacher: The relationships between probability distributions aren't that complicated.

Literally the relationships:



Normal Distribution Idea

The Normal (aka Gaussian) distribution is perhaps the most famous continuous probability distribution. It is often used as the go-to distribution to represent the distribution of distribution to represent the distribution of unknown random variables.

In the real world, we might be trying to model measurement error, or the distribution of scores for an exam. These scenarios are naturally modeled by the normal distribution.

Normal Distribution Definition

For any $\mu, \sigma > \mathbb{R}$ and $\sigma > 0$, a continuous random variable X with pdf

is called a normal random variable with mean parameter μ and variance σ^2, and we write

$X \sim \mathcal{N}(\mu, \sigma^2)$

In the special case where $\mu = 0$ and $\sigma = 1$, X is called a standard normal random variable.

The CDF of the standard normal has a special name, $\Phi(x) \stackrel{\text{def}}{=}$

"phi"

Picture

"68-95-99.7 rule:"

If $X \sim N(\mu, \sigma^2)$,

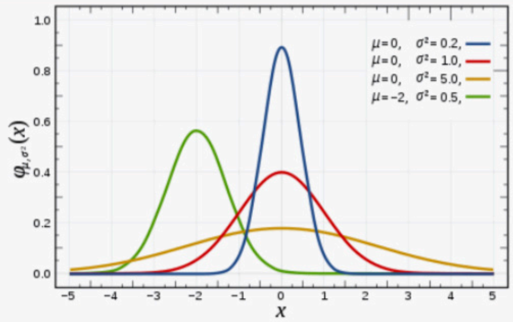
$P(\mu - 1\cdot\sigma \le X \le \mu + 1\cdot\sigma) \approx .68$

$P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx .95$

$P(\mu - 3\sigma \le X \le \mu + 3\sigma) \approx .997$

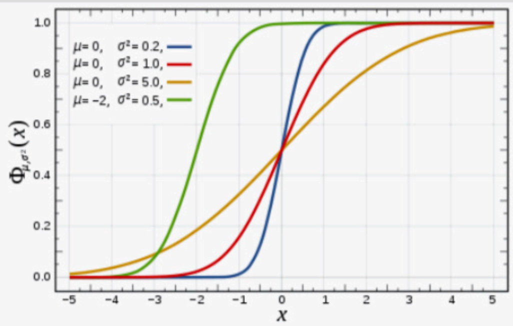
Normal distribution

Probability density function



The red curve is the *standard normal distribution*

Cumulative distribution function



Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbb{R}$ = mean (location) $\sigma^2 > 0$ = variance (squared scale)
Support	$x \in \mathbb{R}$
PDF	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$

no nice closed form

Check.

Is the Normal pdf valid?

↳ Is $f(x)$ nonnegative?

↳ Does pdf integrate to 1?

Mean & Variance of Standard Normal (Check)

$$X \sim \mathcal{N}(0,1)$$

$$\mathbb{E}[X] =$$

$$\mathbb{E}[X^2] =$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 =$$

Scaling & Shifting Normals

Statement: If $X \sim N(\mu, \sigma^2)$, then $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$

Proof:

Let $X \sim N(\mu, \sigma^2)$, we can calculate the distribution of $Y = \frac{X-\mu}{\sigma}$

Mean & Variance of Normal

Let $X \sim N(\mu, \sigma^2)$

What is $\mathbb{E}[X]$? What is $Var[X]$?

We know $Y = \frac{X - \mu}{\sigma}$ is $N(0,1)$

So, for $\mathbb{E}[X]$,

For $Var[X]$,

Relating to Standard Normal (CDF)

We can relate any normal random variable $X \sim N(\mu, \sigma^2)$ to the standard normal Y:

Since the CDF uniquely characterizes a distribution, we can use a table of precomputed values of $\Phi(x)$ for different values of x to do probability computations with normal distributions

Using Table of Precomputed Values (Example)

Suppose $X \sim N(60, 20^2)$, and we want to find $\mathbb{P}(X \geq 80)$

Standard Normal CDF Table

Introduction to Probability, 2nd Ed, by D. Bertsekas and J. Tsitsiklis, Athena Scientific, 2008

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The standard normal table. The entries in this table provide the numerical values of $\Phi(y) = \mathbf{P}(Y \leq y)$, where Y is a standard normal random variable, for y between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01. so that $\Phi(1.71) = .9564$. When y is negative. the value of $\Phi(y)$ can be found using the formula $\Phi(y) = 1 - \Phi(-y)$.

Nice Property: Sum of Independent Gaussians is Gaussian.

If $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_y, \sigma_y^2)$,

then $Z = X + Y$ has distribution

Proof: See note 17, Section 4.2

Central Limit Theorem

For i.i.d random variables X_i each with mean μ and variance σ^2, $S_n = \sum_{i=1}^{n} X_i$

The Central Limit Theorem states:

The CDF of the standardized sample mean of the X_i converges to $\Phi()$ as $n \to \infty$, regardless of the distribution of the X_i so long as their mean and variance are finite.

CLT Example.

Consider a flash drive that stores 1000 bits of information.
Due to natural factors, a bit is corrupted with probability 0.1. Assuming that the corruption for the bits occur independently, what is the probability that there are more than 110 corrupted bits?