1. Joint PDFs

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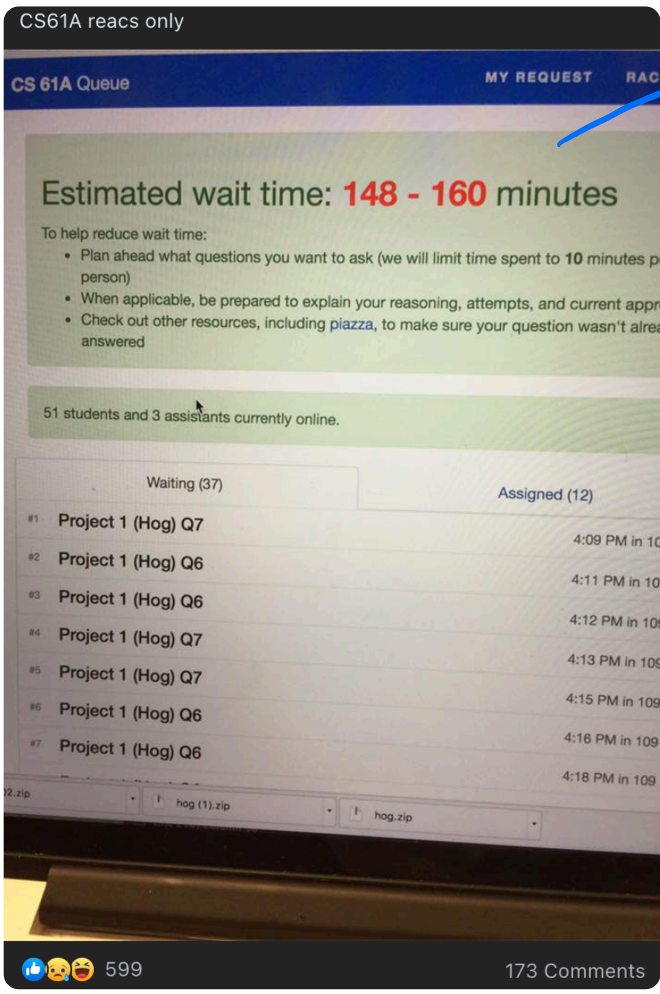
5. Examples



Estimate is from CLT

Theory

Queue Theory

Practice

>120

Joint PDFs

Let X and Y be two continuous random variables.

Then the joint density function $f_{X,Y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

satisfies :

$$\mathbb{P}\left(a \leq X \leq b,\ c \leq Y \leq d\right) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx = 1$$

and

$$f_{X,Y}(x,y) \geq 0 \qquad \forall\, x,y \in \mathbb{R}$$

pdf in 2 variable joint case

"probability per unit area"

Joint CDFs

Let X and Y be two random variables.
Then the joint cumulative distribution function
$F_{X,Y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$

In the continuous case,
to get back to joint pdf

$$f_{X,Y}(x,y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} F_{X,Y}(x,y)$$

$$= \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Conditional PDFs

Let X and Y be two continuous random variables with joint density function $f_{X,Y}$

For any y with $f(y) > 0$, the conditional distribution of X given $Y=y$ is defined as:

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Ex: $Y \sim Exp(\lambda) \rightarrow f_Y(y) = \lambda e^{-\lambda y}$ for $y \geq 0$

When Y is continuous, even though $P(Y=y)=0$,

if $\underline{f_Y(y) > 0}$,

$$\mathbb{P}(a \leq X \leq b \mid Y=y) = \int_a^b f_{X|Y=y}(x)\, dx$$

Independence

Let X and Y be two continuous random variables.
X and Y are independent if:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

for all x, y.

Since $f_{X,Y}(x,y) = f_{X|Y}(x) \cdot f_Y(y)$

this also implies that $\underbrace{f_{X|Y}(x) = f_X(x)}_{\text{Must be true for } X \text{ and } Y \text{ to be independent.}}$

Note:

$f \rightarrow$ pdf

$F \rightarrow$ cdf

Marginalization

To recover the individual pdfs from the $\overbrace{\text{joint pdf}}^{f_{x,y}(x,y)}$:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

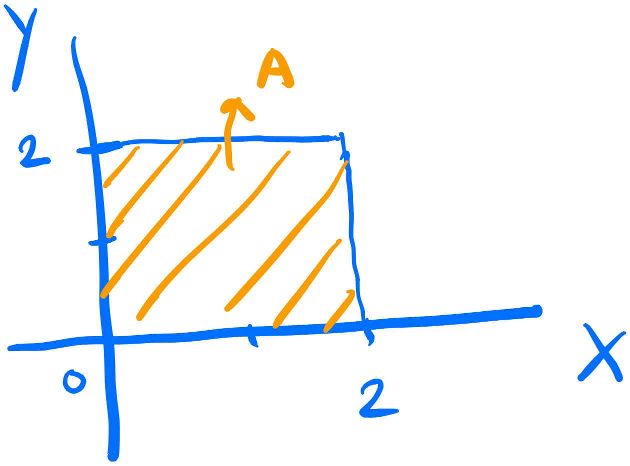
$f_X(x)$ needs to be a valid pdf. (check)

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$$

$$= 1$$ since the joint pdf is valid

Example: $X, Y \overset{iid}{\sim} \text{Unif}(0,2)$

What is $f_{X,Y}(x,y)$ for two uniform r.v.s on $[0,2]$?



$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in A \\ 0 & \text{o.w.} \end{cases}$$

Uniform on 2x2 square $\rightarrow$ constant c

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx = 1$$

$$\int_0^2 \int_0^2 c \; dy\, dx = 1$$

$$c \int_0^2 \boxed{\int_0^2 1 \cdot dy}\, dx = 1$$

$$c \int_0^2 2\, dx$$

$$2 \cdot c \cdot \boxed{\int_0^2 1 \cdot dx}$$

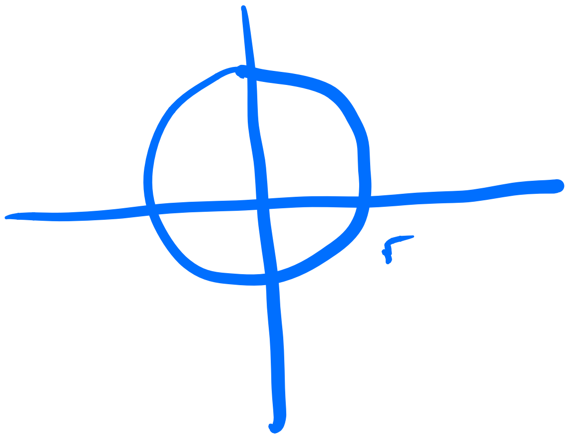
$$2 \cdot c \cdot 2 = 4c$$

$$c \cdot 4 = 1 \Rightarrow c = \frac{1}{4}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4} & \text{if } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

Uniform Density Over a Disk: Joint

What is $f_{X,Y}(x,y)$ for a uniform density over a disk of radius r centered at the origin?



$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{o.w.} \end{cases}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx = \iint_{x^2+y^2\le r^2} c \cdot dx \cdot dy$$

$$= c \iint_{x^2+y^2 \le r} 1 \cdot dx\, dy$$

$$= c \cdot (\text{area of the disk})$$

$$= c \cdot \pi r^2$$

We know the joint pdf must integrate to 1

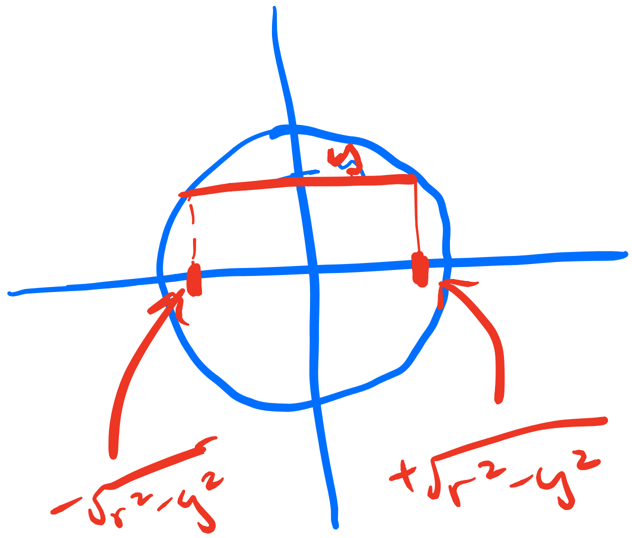
$$c\pi r^2 = 1 \Rightarrow c = \frac{1}{\pi r^2}$$

$$\Rightarrow f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi r^2} & \text{if } x^2+y^2 \le r^2 \\ 0 & \text{o.w.} \end{cases}$$

Uniform Density Over a Disk: Marginals

What is $f_y(y)$ and $f_x(x)$, now that we know the joint over the disk?

$$f_y(y) = \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx$$



$$= \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx$$

$$= \frac{1}{\pi r^2}\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} 1 \cdot dx$$

$x^2 + y^2 \leq r^2$

$x \in \left(-\sqrt{r^2-y^2}, \sqrt{r^2-y^2}\right)$

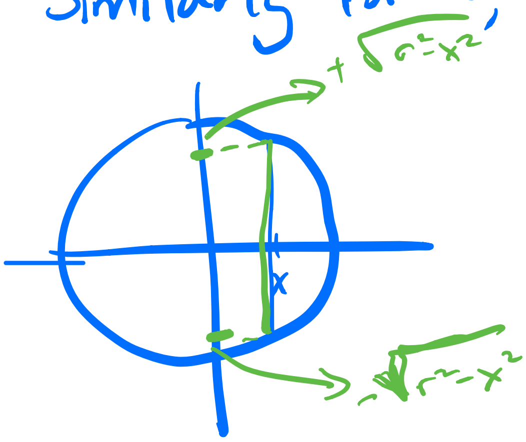
$$= \frac{1}{\pi r^2}\left(2\cdot\sqrt{r^2-y^2}\right)$$

for $-r \leq y \leq r$

0 o.w.

$$\rightarrow f_y(y) = \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & \text{if } -r\leq y\leq r \\ 0 & \text{o.w.} \end{cases}$$

Similarly for x,



$$f_x(x) = \begin{cases} \dfrac{2\sqrt{r^2-x^2}}{\pi r^2} & \text{if } -r\leq x\leq r \\ 0 & \text{o.w.} \end{cases}$$

$x^2+y^2 \leq r^2$ to be on disk.

Uniform Density over a Disk: Conditional PDFs.

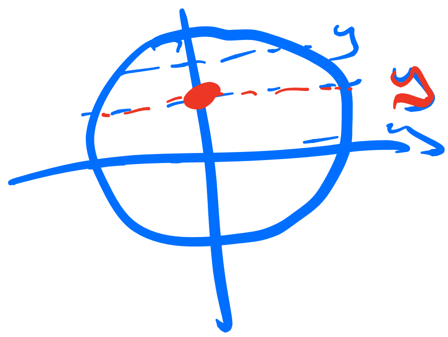
What is $f_{X|Y=y}(x)$?

By def. $f_{X|Y=y} = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{\frac{1}{\pi r^2}}{\frac{2\sqrt{r^2-y^2}}{\pi r^2}}$

Note:

$f_{X|Y=y}(x) \neq \dfrac{2\sqrt{r^2-x^2}}{\pi r^2} = f_X(x)$ $= \dfrac{1}{2\sqrt{r^2-y^2}}$

$\Rightarrow$ X and Y are not independent. for $-r < y < r$



What about $\mathrm{Cov}(X,Y)$?

$\mathbb{E}[X] = 0$

$\mathbb{E}[Y] = 0$

$\mathbb{E}[X|Y=y] = 0$ $\quad \mathbb{E}[X\cdot y | Y=y]$

$\mathbb{E}[X\cdot Y | Y=y] = y\,\mathbb{E}[X|Y=y] = y\cdot 0 = 0$

$\Rightarrow \mathbb{E}[XY] = 0$

$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
$= 0 - 0 = 0$

$\Rightarrow$ X, Y are uncorrelated (0 Covariance), but are dependent.

Independent $\Rightarrow$ uncorrelated, uncorrelated ~~$\Rightarrow$~~ independent

2D LOTUS

BREAK: 4:10 pm

Let X, Y be two random variable with joint PDF $f_{X,Y}(x,y)$, and let $g(x,y)$ be a real-valued function of x, y. Then,

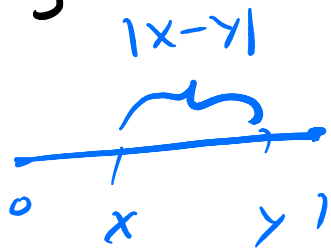
$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) \cdot f_{X,Y}(x,y)\,dy\,dx$$

using the known joint pdf of X and Y

You don't need to calculate the pdf of $g(X,Y)$ to calculate its expectation if you have $f_{X,Y}(x,y)$

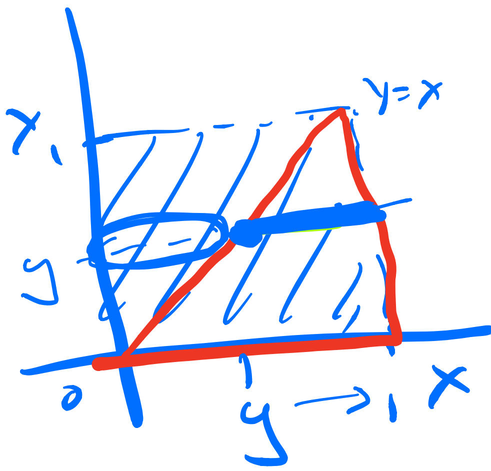
Expected Distance Between Two Points

Let $X, Y \overset{i.i.d}{\sim} \text{Unif}(0,1)$. What is $\mathbb{E}[|X-Y|]$?



$$\mathbb{E}[|X-Y|] = \int_0^1 \int_0^1 |x-y| f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 |x-y| \cdot 1 \cdot dx\, dy$$



$$= \iint_{\boxed{x>y}} (x-y)\, dx\, dy + \iint_{y>x} (y-x)\, dx\, dy$$

$$= 2 \cdot \iint_{x>y} (x-y)\, dx\, dy$$

$$= 2 \int_0^1 \int_y^1 (x-y)\, dx\, dy$$

$$= 2 \int_0^1 \left(\frac{x^2}{2} - yx \right) \Bigg|_{x=y}^{x=1} dy$$

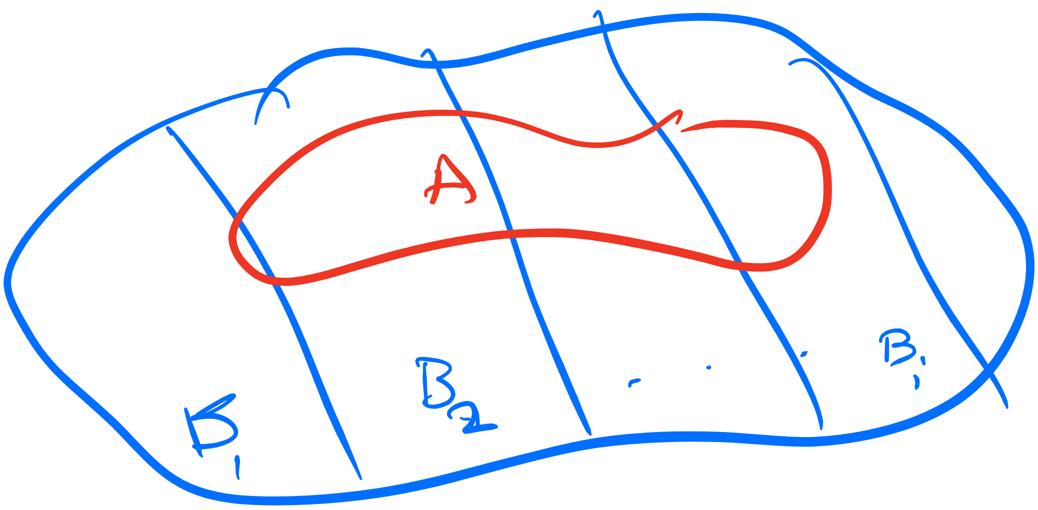
$$= 2 \int_0^1 \left(\frac{y^2}{2} - y + \frac{1}{2} \right) dy = \frac{1}{3}$$

Note:

See full notes

Total Probability Theorem

	Discrete Partition "B_i"	Continuous Partition "B_i"
Discrete "A"	$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(B_i) \cdot \mathbb{P}(A \mid B_i)$	$\mathbb{P}(A) = \int_{-\infty}^{\infty} f_X(x) \cdot \mathbb{P}(A \mid X = x)\, dx$ ①
Continuous "A"	$f_X(x) = \sum_{i=1}^{n} \mathbb{P}(B_i) f_{X\mid B_i}(x)$ ②	$f_X(x) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_{X\mid Y=y}(x)\, dy$



Total Probability Theorem Examples

1. X is continuous
   Y is discrete

$\Rightarrow \mathbb{P}(Y>x) = \int_{-\infty}^{\infty} f_X(x) \cdot \mathbb{P}(Y>x \mid X=x)\, dx$

2. Y is coin flip.

$X \sim \text{Exp}(\lambda_1)$ if Y is heads $\leftarrow f_{X|B_1}$

$X \sim \text{Exp}(\lambda_2)$ if Y is tails $\leftarrow f_{X|B_2}$

Then for $x>0$,

$$f_X(x) = \frac{1}{2} \cdot \lambda_1 e^{-\lambda_1 x} + \frac{1}{2} \cdot \lambda_2 e^{-\lambda_2 x}$$

Bayes' Rule "$\mathbb{P}(A|B)$"

	Discrete "B"	Continuous "B"
Discrete "A"	$P(A_i\|B) = \dfrac{P(A_i)P(B\|A_i)}{\sum_{j=1}^{n} P(A_j)P(B\|A_j)}$	$\mathbb{P}(A_i\|X=x) = \dfrac{\mathbb{P}(A_i)\cdot f_{X\|A_i}(x)}{\sum_{j=1}^{n} P(A_j) f_{X\|A_j}(x)}$
Continuous "A"	$f_{X\|B}(x) = \dfrac{f_X(x)\cdot \mathbb{P}(B\|X=x)}{\int_{-\infty}^{\infty} f_X(t)\cdot \mathbb{P}(B\|X=t)\,dt}$	$f_{X\|Y}(x) = \dfrac{f_X(x)\cdot f_{Y\|X=x}(y)}{\int_{-\infty}^{\infty} f_X(t)\cdot f_{Y\|X=t}(y)\,dt}$

Example:
(Discrete/ Continuous)

See Full Notes