Review (Basics)

Def A Markov chain is

In this class, a Markov chain is specified by:

Thm Let $\mathbb{P}_n$ be the n-step transition matrix $\mathbb{P}_n(i,j) = \mathbb{P}(X_n = j \mid X_0 = i)$
π_n be the distribution over the states after n steps.

Then

Review (Conditioning)

<u>Thm</u> Let $A, B \subset S$ be mutually exclusive. Let T_A, T_B be the times until we visit a state in A and B, respectively.
Define $\alpha(i) = \mathbb{P}(T_A < T_B \mid X_0 = i)$, $\beta(i) = \mathbb{E}[T_A \mid X_0 = i]$
Then

<u>Thm</u> Let V_A be the number of times a state in A is visited.
Define $\gamma(i) = \mathbb{E}[V_A \mid X_0 = i]$.
Then

Compartmental Model

Ex Suppose that for some disease, the population can be split into three groups: susceptible, infected, and removed.

Each day,

- of the susceptible, 20% become infected;
  80% stay susceptible.
- of the infected, 20% become susceptible;
  10% become removed;
  70% stay infected
- of the removed, p become susceptible
  $1-p$ stay removed.

In the population prior to exposure, 95% are susceptible and 5% are removed.

Invariant Distributions

Def A distribution π

Thm $\pi_n = \pi_0$ for all $n \geq 0$ if and only if π_0 is invariant

Pf

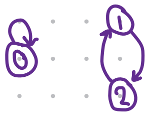
Note A Markov chain may have many invariant distributions. For example, $\mathbb{P} = \mathbb{I}$ has infinitely many.

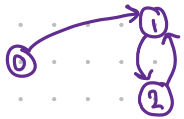
Note Invariance means that the net flow in and out of states is equal.

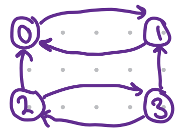
Irreducibility

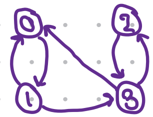
Def We say

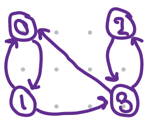
Ex Which of the following chains is irreducible?











Long Run Proportion of Time

<u>Thm</u> If a finite Markov chain is irreducible, then, for any initial distribution π_0,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \mathbb{1}\{X_m = i\} = \pi(i) \quad \text{for all } i \in S$$

That is, the fraction of time spent in each state is given by π.
Also, π is invariant; therefore the invariant distribution exists and is unique

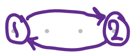
<u>Pf</u>

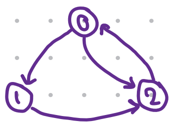
<u>Note</u> An irreducible Markov chains distribution does not necessarily converge to π:

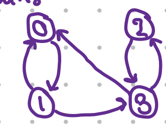
Periodicity

Def

Ex Find $d(\cdot)$ for each of the following chains





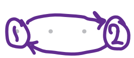


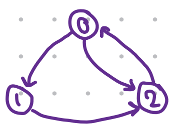
Clm If a Markov chain is irreducible,

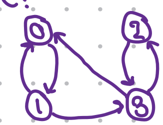
Pf As an exercise

Def If

Ex Which of the following chains is aperiodic?







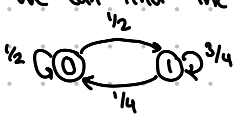
Markov Chain Convergence Theorem

Thm (Finite Markov Chain Convergence) If $X_0, X_1, \ldots$ is a Markov chain on S with time-homogeneous transition matrix $\mathbb{P}$ and

Cor $\pi_n(i) = \mathbb{P}(X_n = i) \to \pi(i)$ as $n \to \infty$

Pf

Note We can find the long-run probability of an event by conditioning on π.



$\pi = [\frac{1}{3}, \frac{2}{3}]$

What is the long-run probability that we stay in the same state?

Ehrenfest Chain I

Q In the Ehrenfest model, there are two containers, containing a total of N particles.
At each step
- a container is selected uniformly at random
- a particle is selected uniformly at random, independently of the container

and the selected particle is placed in the selected container; if the particle was already in the container, it remains in place.
Let X_n be the number of particles in the first container at time n.
What are the transition probabilities of the chain?

Prove that any distribution over the states converges to some distribution π.

Ehrenfest Chain II

Q In the Ehrenfest model, there are two containers, containing a total of N particles.
At each step
- a container is selected uniformly at random
- a particle is selected uniformly at random, independently of the container

and the selected particle is placed in the selected container; if the particle was already in the container, it remains in place.
Let X_n be the number of particles in the first container at time n.
Find the stationary distribution of the chain.